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MIRZAKHARNI'S RECURSION FORMULA IS EQUIVALENT TO THE WITTEN-KONTSEVICH THEOREM

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper, we give a proof of Mirzakhani's recursion formula of Weil-Petersson volumes of moduli spaces of curves using the Witten-Kontsevich theorem. We also describe properties of intersections numbers involving higher degree κ classes.

***Résumé* (La formule de récurrence de Mirzakhani est équivalente au théorème de Witten-Kontsevich)**

Dans cet article, nous démontrons la formule de récurrence de Mirzakhani sur les volumes de Weil-Petersson des espaces de module de courbes en utilisant le théorème de Witten-Kontsevich. Nous donnons aussi des propriétés des nombres d'intersection associées aux classes κ de degré supérieur.

1. Introduction

Following the notation of Mulase and Safnuk [**21**], let $\mathscr{M}_{g,n}(\mathbf{L})$ denote the moduli space of bordered Riemann surfaces with n geodesic boundary components of specified lengths $\mathbf{L} = (L_1, \dots, L_n)$ and let $\mathrm{Vol}_{g,n}(\mathbf{L})$ denote its Weil-Petersson volume $\mathrm{Vol}(\mathscr{M}_{g,n}(\mathbf{L}))$. Using her remarkable generalization of the McShane identity, Mirzakhani [**19**] proved a beautiful recursion formula for these Weil-Petersson volumes

$$\begin{aligned}\mathrm{Vol}_{g,n}(\mathbf{L}) = \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ \underline{n}=I\coprod J}} \int_0^{L_1}\int_0^{\infty}\int_0^{\infty} & xyH(t,x+y) \\ & \times \mathrm{Vol}_{g_1,n_1}(x,\mathbf{L}_I)\mathrm{Vol}_{g_2,n_2}(y,\mathbf{L}_J)dxdydt \\ + \frac{1}{2L_1}\int_0^{L_1}\int_0^{\infty}\int_0^{\infty} xyH(t,x+y)&\mathrm{Vol}_{g-1,n+1}(x,y,L_2,\dots,L_n)dxdydt\end{aligned}$$

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$$+\frac{1}{2L_1}\sum_{j=2}^{n}\int_0^{L_1}\int_0^{\infty} x\big(H(x,L_1+L_j)+H(x,L_1-L_j)\big)$$
$$\times \mathrm{Vol}_{g,n-1}(x,L_2,\dots,\hat{L_j},\dots,L_n)dxdt,$$

where the kernel function

$$H(x,y)=\frac{1}{1+e^{(x+y)/2}}+\frac{1}{1+e^{(x-y)/2}}.$$

Using symplectic reduction, Mirzakhani [20] showed the following relation

$$\frac{\mathrm{Vol}_{g,n}(2\pi\mathbf{L})}{(2\pi^2)^{3g+n-3}}=\frac{1}{(3g+n-3)!}\int_{\mathcal{M}_{g,n}}(\kappa_1+\sum_{i=1}^{n}L_i^2\psi_i)^{3g+n-3}$$
$$=\sum_{\substack{d_0+\dots+d_n\\=3g+n-3}}\prod_{i=0}^{n}\frac{1}{d_i!}\langle\kappa_1^{d_0}\prod\tau_{d_i}\rangle_{g,n}\prod_{i=1}^{\infty}L_i^{2d_i}.$$

Combining with her recursion formula of Weil-Petersson volumes, Mirzakhani [20] found a new proof of the celebrated Witten-Kontsevich theorem.

By taking derivatives with respect to $\mathbf{L}=(L_1,\dots,L_n)$ in Mirzakhani's recursion, Mulase and Safnuk [21] obtained the following enlightening recursion formula of intersection numbers which is equivalent to Mirzakhani's recursion.

$$(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^a\rangle_g$$
$$=\sum_{j=2}^{n}\sum_{b=0}^{a}\frac{a!}{(a-b)!}\frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!}\beta_b\langle\kappa_1^{a-b}\tau_{b+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$
$$+\frac{1}{2}\sum_{b=0}^{a}\sum_{r+s=b+d_1-2}\frac{a!}{(a-b)!}(2r+1)!!(2s+1)!!\beta_b\langle\kappa_1^{a-b}\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{b=0}^{a}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\dots,n\}}}\sum_{r+s=b+d_1-2}\frac{a!}{c!c'!}(2r+1)!!(2s+1)!!\beta_b$$
$$\times\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},$$

where

$$\beta_b=(2^{2b+1}-4)\frac{\zeta(2b)}{(2\pi^2)^b}=(-1)^{b-1}2^b(2^{2b}-2)\frac{B_{2b}}{(2b)!}.$$

Safnuk [23] gave a proof of the above differential form of Mirzakhani's recurson formula using localization techniques, but he also used the Mirzakhani-McShane formula. The relationship between Mirzakhani's recurson and matrix integrals has been studied by Eynard-Orantin [7] and Eynard [6].

Indeed, when $a = 0$, Mulase-Safnuk differential form of Mirzakhani's recursion is just the Witten-Kontsevich theorem [**14, 24**] in the form of DVV recursion relation [**4**]. There are several other new proofs of Witten-Kontsevich theorem [**3, 12, 13, 22**] besides Mirzakhani's proof [**20**].

More discussions about Weil-Petersson volumes from the point of view of intersection numbers can be found in the papers [**5, 10, 18, 26**].

In Section 2, we show that Mirzakhani's recursion formula is essentially equivalent to the Witten-Kontsevich theorem via a formula from [**11**] expressing κ classes in terms of ψ classes. In Section 3, we present certain results of intersection numbers involving higher degree κ classes.

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2. Proof of Mirzakhani's recursion formula

We first give three lemmas. The following lemma can be found in [**21**].

Lemma 2.1. — *The constants β_b in Mirzakhani's recursion satisfy the following:*

$$\sum_{k=0}^{\infty} \beta_k x^k = \frac{\sqrt{2x}}{\sin\sqrt{2x}}.$$

And its inverse:

$$\Big(\sum_{k=0}^{\infty} \beta_k x^k\Big)^{-1} = \frac{\sin\sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{(2k+1)!} x^k.$$

Proof. — Since

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n} = \frac{x}{2}\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \frac{x}{2i}\cot\frac{x}{2i},$$

we have

$$\sum_{k=0}^{\infty} \beta_k x^k = \sqrt{2x}\Big(\cot\sqrt{\frac{x}{2}} - \cot\sqrt{2x}\Big) = \frac{\sqrt{2x}}{\sin\sqrt{2x}}. \qquad \square$$

The following elementary result is crucial to our proof.

Lemma 2.2. — *Let $F(m,n)$ and $G(m,n)$ be two functions defined on $\mathbb{N}\times\mathbb{N}$, where $\mathbb{N} = \{0,1,2,\ldots\}$ is the set of nonnegative integers. Let α_k and β_k be real numbers that satisfy*

$$\sum_{k=0}^{\infty} \alpha_k x^k = \Big(\sum_{k=0}^{\infty} \beta_k x^k\Big)^{-1}.$$

Then the following two identities are equivalent:

$$G(m,n) = \sum_{k=0}^{m} \alpha_k F(m-k, n+k), \quad \forall\ (m,n)\in\mathbb{N}\times\mathbb{N},$$

$$F(m,n)=\sum_{k=0}^{m}\beta_k G(m-k,n+k),\quad \forall\ (m,n)\in\mathbb{N}\times\mathbb{N}.$$

Proof. — Assume the first identity holds, then we have

$$\begin{aligned}\sum_{i=0}^{m}\beta_i G(m-i,n+i)&=\sum_{i=0}^{m}\beta_i\sum_{j=0}^{m-i}\alpha_j F(m-i-j,n+i+j)\\&=\sum_{k=0}^{m}\sum_{i+j=k}(\beta_i\alpha_j)F(m-k,n+k)\\&=\sum_{k=0}^{m}\delta_{k0}F(m-k,n+k)\\&=F(m,n).\end{aligned}$$

So we proved the second identity. The proof of the other direction is the same. □

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [9], and has been developed by Arbarello-Cornalba [**2**], Faber [**8**] and Kaufmann-Manin-Zagier [**11**] into a nice combinatorial formalism.

Lemma 2.3 **([11])**. — *For* $m>0$,

$$\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^m\rangle_g=\sum_{k=1}^{m}\frac{(-1)^{m-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=m\\ m_i>0}}\binom{m}{m_1,\ldots,m_k}\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{j=1}^{k}\tau_{m_j+1}\rangle_g.$$

Proof. — (sketch) Let $\pi_{n+p,n}:\overline{\mathcal{M}}_{g,n+p}\longrightarrow\overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1}\cdots\psi_{n+p}^{a_p+1})$ by $R(a_1,\ldots,a_p)$, then we have the formula from [**2**]

$$R(a_1,\ldots,a_p)=\sum_{\sigma\in\mathbb{S}_p}\prod_{\substack{\text{each cycle } c\\ \text{of } \sigma}}\kappa_{\sum_{j\in c}a_j},$$

where we write any permutation σ in the symmetric group $\mathbb{S}_p$ as a product of disjoint cycles.

A formal combinatorial argument [**11**] leads to the following inversion equation

$$\kappa_{a_1}\cdots\kappa_{a_p}=\sum_{k=1}^{p}\frac{(-1)^{p-k}}{k!}\sum_{\substack{\{1,\ldots,p\}=S_1\coprod\cdots\coprod S_k\\ S_k\neq\varnothing}}R(\sum_{j\in S_1}a_j,\ldots,\sum_{j\in S_k}a_j),$$

from which the result follows easily. □

Proposition 2.4. — *We have*

$$\sum_{b=0}^{a}(-1)^b\binom{a}{b}\frac{(2(d_1+b)+1)!!}{(2b+1)!!}\langle\tau_{d_1+b}\prod_{i=2}^{n}\tau_{d_i}\kappa_1^{a-b}\rangle_g$$
$$=\sum_{j=2}^{n}\frac{(2d_1+2d_j-1)!!}{(2d_j-1)!!}\langle\kappa_1^a\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\kappa_1^a\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{c+c'=a\\ I\coprod J=\{2,\dots,n\}}}\binom{a}{c}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\kappa_1^c\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.$$

Proof. — Let LHS and RHS denote the left and right hand side of the equation respectively. By Lemma 2.3 and the Witten-Kontsevich theorem, we have

$$(2d_1+1)!!\langle\prod_{j=1}^{n}\tau_{d_j}\kappa_1^a\rangle_g$$
$$=(2d_1+1)!!\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\dots,m_k}\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{j=1}^{k}\tau_{m_j+1}\rangle_g$$
$$=\sum_{k=0}^{a}\frac{(-1)^{a-k}}{k!}\sum_{\substack{m_1+\cdots+m_k=a\\ m_i>0}}\binom{a}{m_1,\dots,m_k}$$
$$\times\Bigg(\sum_{j=2}^{n}\frac{(2(d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\tau_{d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_g$$
$$+\sum_{j=1}^{k}\frac{(2(d_1+m_j)+1)!!}{(2m_j+1)!!}\langle\tau_{d_1+m_j}\prod_{i=2}^{n}\tau_{d_i}\prod_{i\neq j}\tau_{m_i+1}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\prod_{i=2}^{n}\tau_{d_i}\prod_{i=1}^{k}\tau_{m_i+1}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{I\coprod J=\{2,\dots,n\}\\ I'\coprod J'=\{1,\dots,k\}}}\sum_{r+s=d_1-2}(2r+1)!!(2s+1)!!$$
$$\times\langle\tau_r\prod_{i\in I}\tau_{d_i}\prod_{i\in I'}\tau_{m_i+1}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\prod_{i\in J'}\tau_{m_i+1}\rangle_{g-g'}\Bigg)$$