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THE INDEX OF PROJECTIVE FAMILIES OF ELLIPTIC OPERATORS: THE DECOMPOSABLE CASE

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — An index theory for projective families of elliptic pseudodifferential operators is developed under two conditions. First, that the twisting, i.e. Dixmier-Douady, class is in $\mathrm{H}^2(X;\mathbb{Z})\cup\mathrm{H}^1(X;\mathbb{Z})\subset\mathrm{H}^3(X;\mathbb{Z})$ and secondly that the 2-class part is trivialized on the total space of the fibration. One of the features of this special case is that the corresponding Azumaya bundle can be refined to a bundle of smoothing operators. The topological and the analytic index of a projective family of elliptic operators associated with the smooth Azumaya bundle both take values in twisted K-theory of the parameterizing space and the main result is the equality of these two notions of index. The twisted Chern character of the index class is then computed by a variant of Chern-Weil theory.

Résumé **(L'indice des familles projectives d'opérateurs elliptiques: le cas décomposable)**
Une théorie de l'indice pour des familles projectives d'opérateurs pseudodifférentiels elliptiques est développée sous les deux conditions suivantes: la classe de Dixmier-Douady est dans $\mathrm{H}^2(X;\mathbb{Z})\cup\mathrm{H}^1(X;\mathbb{Z})\subset\mathrm{H}^3(X;\mathbb{Z})$, et la partie de degré deux est trivialisée sur l'espace total de la fibration. Le fibré d'Azumaya correspondant peut alors être raffiné en un fibré d'opérateurs régularisants. Les indices topologiques et analytiques d'une famille projective d'opérateurs elliptiques associée au fibré d'Azumaya lisse sont à valeurs dans la K-théorie tordue de la base de la famille et le résultat principal est l'égalité de ces deux indices. Le caractère de Chern tordu de la famille est calculé par une variante de la théorie de Chern-Weil.

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Introduction

The basic object leading to twisted K-theory for a space, X, can be taken to be a principal PU-bundle $\mathscr{P} \longrightarrow X$, where $\mathrm{PU} = \mathrm{U}(\mathscr{H})/\mathrm{U}(1)$ is the group of projective unitary operators on some separable infinite-dimensional Hilbert space $\mathscr{H}$. Circle bundles over X are classified up to isomorphism by their Chern classes in $\mathrm{H}^2(X;\mathbb{Z})$ and analogously principal PU bundles are classified by $\mathrm{H}^3(X;\mathbb{Z})$ with the element $\delta(\mathscr{P})$ being the Dixmier-Douady invariant of $\mathscr{P}$. Just as $\mathrm{K}^0(X)$, the ordinary K-theory group of X, may be identified with the group of homotopy classes of maps $X \longrightarrow \mathscr{F}(\mathscr{H})$ into the Fredholm operators on $\mathscr{H}$, the twisted K-theory group $\mathrm{K}^0(X;\mathscr{P})$ may be identified with the homotopy classes of sections of the bundle $\mathscr{P} \times_{\mathrm{PU}} \mathscr{F}$ arising from the conjugation action of PU on $\mathscr{F}$. The action of PU on the compact operators, $\mathscr{K}$, induces the *Azumaya* bundle, $\mathscr{A}$. The K-theory, in the sense of C^* algebras, of the space of continuous sections of this bundle, written $\mathrm{K}^0(X;\mathscr{A})$, is naturally identified with $\mathrm{K}^0(X;\mathscr{P})$. From an analytic viewpoint $\mathscr{A}$ is more convenient to deal with than $\mathscr{P}$ itself.

In the case of circle bundles isomorphisms are classified up to homotopy by an element of $\mathrm{H}^1(X;\mathbb{Z})$, corresponding to the homotopy class of a smooth map $X \longrightarrow \mathrm{U}(1)$. Similarly, $\delta \in \mathrm{H}^3(X;\mathbb{Z})$ determines $\mathscr{P}$ up to isomorphism with the isomorphism class determined up to homotopy by an element of $\mathrm{H}^2(X;\mathbb{Z})$, corresponding to the fact that PU is a $K(\mathbb{Z},2)$. The result is that $\mathrm{K}^0(X;\mathscr{A})$ depends as a group on the choice of Azumaya bundle with DD invariant δ up to an action of $\mathrm{H}^2(X;\mathbb{Z})$.

In **[20]** we extended the index theorem for a family of elliptic operators, giving the equality of the analytic and the topological index maps in K-theory, to the case of twisted K-theory where the twisting class is a torsion element of $\mathrm{H}^3(X;\mathbb{Z})$. In this paper we prove a similar index equality in the case of twisted K-theory when the index class is decomposable

$$\delta = \alpha \cup \beta,\ \alpha \in \mathrm{H}^1(X;\mathbb{Z}),\ \beta \in \mathrm{H}^2(X;\mathbb{Z}), \tag{1}$$

and the fibration $\phi : Y \longrightarrow X$ is such that $\phi^*\beta = 0$ in $\mathrm{H}^2(Y;\mathbb{Z})$.

Under the assumption (1), that the class δ is *decomposed,* we show below that there is a choice of principal PU bundle with class δ such that the classifying map above, $c_P : X \longrightarrow K(\mathbb{Z};3)$ factors through $\mathrm{U}(1) \times \mathrm{PU}$. Twisting by a homotopically non-trivial map $\kappa : X \longrightarrow \mathrm{PU}$ does not preserve this property, so in this decomposed case there is indeed a natural choice of smooth Azumaya bundle, $\mathscr{J}$, up to homotopically trivial isomorphism and this induces a choice of twisted K-group determined by the decomposition of δ; we denote this well-defined twisted K-group by

$$\mathrm{K}^0(X;\alpha,\beta) = \mathrm{K}^0(X;\mathscr{A}), \quad \mathscr{A} = \overline{\mathscr{J}}. \tag{2}$$

The effect on smoothness of the assumption of decomposability on the Dixmier-Douady class can be appreciated by comparison with the simpler case of degree 2. Thus, if $\alpha_1 \cup \alpha_2 \in \mathrm{H}^2(X;\mathbb{Z})$ is a decomposed class, $\alpha_i \in \mathrm{H}^1(X;\mathbb{Z})$ for $i = 1, 2$, then the associated line bundle is the pull-back of the Poincaré line bundle associated to a polarization on the 2-torus under the map $u_1 \times u_2$, where the $u_i \in \mathscr{C}^\infty(X;\mathrm{U}(1))$

represent the α_i. This is to be contrasted with the general case in with the line bundle is the pull-back from a classifying space such as PU, and is only unique up to twisting by a smooth map $\kappa' : X \longrightarrow \mathrm{U}(1)$.

The data we use to define a smooth Azumaya bundle is:

– A smooth function

$$u \in \mathscr{C}^\infty(X; \mathrm{U}(1)) \tag{3}$$

the homotopy class of which represents $\alpha \in \mathrm{H}^1(X, \mathbb{Z})$.

– A Hermitian line bundle (later with unitary connection)

$$\begin{array}{c} L \\ \big\downarrow \\ X \end{array} \tag{4}$$

with Chern class $\beta \in \mathrm{H}^2(X; \mathbb{Z})$.

– A smooth fiber bundle of compact manifolds

$$\begin{array}{ccc} Z & \text{——} & Y \\ & & \big\downarrow \phi \\ & & X \end{array} \tag{5}$$

such that $\phi^*\beta = 0$ in $\mathrm{H}^2(Y; \mathbb{Z})$.

– An explicit global unitary trivialization

$$\gamma : \phi^*(L) \xrightarrow{\simeq} Y \times \mathbb{C}. \tag{6}$$

These hypotheses are satisfied by taking $Y = \tilde{L}$, the circle bundle of L, and then there is a natural choice of γ in (6). This corresponds to the 'natural' smooth Azumaya bundle associated to the given decomposition of $\delta = \alpha \cup \beta$ and we take $\mathrm{K}^0(X; \alpha, \beta)$ in (2) to be defined by this Azumaya bundle, discussed as a warm-up exercise in Section 1. In Appendix C it is observed that any fibration for which β is a multiple of a degree 2 characteristic class of $\phi : Y \longrightarrow X$ satisfies the hypothesis in (5).

In general, the data (3) – (6) are shown below to determine an infinite rank 'smooth Azumaya bundle', which we denote $\mathcal{S}(\gamma)$. This has fibres isomorphic to the algebra of smoothing operators on the fibre, Z, of Y with Schwartz kernels consisting of the smooth sections of a line bundle $J(\gamma)$ over Z^2. The completion of this algebra of 'smoothing operators' to a bundle with fibres modelled on the compact operators has Dixmier-Douady invariant $\alpha \cup \beta$.

In outline the construction of $\mathcal{S}(\gamma)$ proceeds as follows; details may be found in Section 3. The trivialization (6) induces a groupoid character $Y^{[2]} \longrightarrow \mathrm{U}(1)$, where $Y^{[2]}$ is the fiber product of two copies of fibration. Combined with the choice (3) this gives a map from $Y^{[2]}$ into the torus and hence by pull-back the line bundle $J = J(\gamma)$. This line bundle is *primitive* in the sense that under lifting by the three projection maps

$$\tilde{L}^{[3]} \overset{\pi_S}{\underset{\pi_F}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\pi_C}}}}} \tilde{L}^{[2]} \tag{7}$$

(corresponding respectively to the left two, the outer two and the right two factors) there is a natural isomorphism

$$\pi_S^* J \otimes \pi_F^* J = \pi_C^* J. \tag{8}$$

This is enough to give the space of global sections, $\mathcal{C}^\infty(Y^{[2]}; J \otimes \Omega_R)$, where Ω_R is the fiber-density bundle on the right factor, a fibrewise product isomorphic to the smoothing operators on Z. Indeed, if z represents a fiber variable then

$$A \circ B(x, z, z') = \int_Z A(x, z, z'') \cdot B(x, z'', z') \tag{9}$$

where $\cdot$ denotes the isomorphism (8) which gives the identification

$$J_{(z,z'')} \otimes J_{(z'',z')} \simeq J_{(z,z')} \tag{10}$$

needed to interpret the integral in (9). The naturality of the isomorphism corresponds to the associativity of this product.

Then the smooth Azumaya bundle is defined in terms of its space of global sections

$$\mathcal{C}^\infty(X; \mathcal{S}(\gamma)) = \mathcal{C}^\infty(Y^{[2]}; J(\gamma)). \tag{11}$$

As remarked above, $J(\gamma)$, and hence also the Azumaya bundle, depends on the particular global trivialization (6). Two trivializations, γ_i, $i = 1, 2$ as in (6) determine

$$\gamma_{12} : Y \longrightarrow \mathrm{U}(1),\ \gamma_{12}(y)\gamma_2(y) = \gamma_1(y) \tag{12}$$

which fixes an element $[\gamma_{12}] \in \mathrm{H}^1(Y; \mathbb{Z})$ and hence a line bundle K_{12} over Y with Chern class $[\gamma_{12}] \cup [\phi^*\alpha]$. Then

$$J(\gamma_2) \simeq (K_{12}^{-1} \boxtimes K_{12}) \otimes J(\gamma_1) \tag{13}$$

with the isomorphism consistent with primitivity.

Pulling back to Y, $\phi^* \mathcal{A}(\gamma)$ is trivialized as an Azumaya bundle and this trivialization induces an isomorphism of twisted and untwisted K-theory

$$\mathrm{K}^0(Y; \phi^* \mathcal{A}(\gamma)) \xrightarrow{\simeq} \mathrm{K}^0(Y). \tag{14}$$

In fact there are stable isomorphisms between the different smooth Azumaya bundles and these induce natural and consistent isomorphisms

$$\mathrm{K}^0(X; \mathcal{A}(\gamma)) \xrightarrow{\simeq} \mathrm{K}^0(X; \alpha, \beta). \tag{15}$$

The proof may be found in Section 4.

The transition maps for the local presentation of the smooth Azumaya bundle, $\mathcal{S}(\gamma)$, determined by the data (3) – (6), are given by multiplication by smooth functions. Thus they also preserve the corresponding spaces of differential, or pseudodifferential, operators on the fibres; the corresponding algebras of twisted fibrewise pseudodifferential operators are therefore well defined. Moreover, since the principal symbol of a pseudodifferential operator is invariant under conjugation by (non-vanishing) functions there is a well-defined symbol map from the pseudodifferential extension of the Azumaya bundle, with values in the usual symbol space on $T^*(Y/X)$ (so with no additional twisting). The trivialization of the Azumaya bundle over Y, and hence over $T^*(Y/X)$, means that the class of an elliptic element can also be interpreted as an element of $\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma))$ where $\rho : T(Y/X) \longrightarrow Y$ is the bundle projection. This leads to the analytic index map,

$$\mathrm{ind}_{\mathrm{a}} : \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma)) \longrightarrow \mathrm{K}^0(X;\mathcal{A}(\gamma)). \tag{16}$$

The topological index can be defined using the standard argument by embedding of the fibration Y into the product fibration $\pi : \mathbb{R}^N \times X \longrightarrow X$ for large N. Namely, the Azumaya bundle is trivialized over Y and this trivialization extends naturally to a fibred collar neighborhood Ω of Y embedded in $\mathbb{R}^N \times X$. Thus, the usual Thom map $\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/X)$ is trivially lifted to a map for the twisted K-theory, which then extends by excision to a map giving the topological index as the composite with Bott periodicity:

$$\begin{aligned}\mathrm{ind}_{\mathrm{t}} : \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma)) &\longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/X);\rho^*\tilde{\pi}^*\mathcal{A}(\gamma)) \\ &\longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^N/X);\rho^*\pi^*\mathcal{A}(\gamma)) \longrightarrow \mathrm{K}^0(X;\mathcal{A}(\gamma)).\end{aligned} \tag{17}$$

In the proof of the equality of these two index maps we pass through an intermediate step using an index map given by semiclassical quantization of smoothing operators, rather than standard pseudodifferential quantiztion. This has the virtue of circumventing the usual problems with multiplicativity of the analytic index even though it is somewhat less familiar. A fuller treatment of this semiclassical approach can be found in **[21]** so only the novelties, such as they are, in the twisted case are discussed here. The more conventional route, as used in **[20]**, is still available but is technically more demanding. In particular it is worth noting that the semiclassical index map, as defined below, is well-defined even for a general fibration – without assuming that $\phi^*\beta = 0$. Indeed, this is essential in the proof, since the product fibration $\mathbb{R}^N \times X$ does not have this property.

For a fixed fibration the index maps induced by two different trivializations γ may be compared and induce a commutative diagram

$$\begin{array}{ccccc}
\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) & \xrightarrow{\simeq} & \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma_1)) & \xrightarrow{\mathrm{ind}(\gamma_1)} & \mathrm{K}^0(X;\mathcal{A}(\gamma_1)) \\
\Big\downarrow [K_{12}]\times & & & & \Big\downarrow \simeq \\
 & & & & \mathrm{K}^0(X;\alpha,\beta) \\
 & & & & \Big\uparrow \simeq \\
\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) & \xrightarrow{\simeq} & \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma_2)) & \xrightarrow{\mathrm{ind}(\gamma_2)} & \mathrm{K}^0(X;\mathcal{A}(\gamma_2)).
\end{array} \tag{18}$$

This follows from the proof of the index theorem.