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DIRECT IMAGE FOR SOME SECONDARY K-THEORIES

by

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This article is dedicated to J.-M. Bismut, for his sixtieth birthday

Abstract. — The real counterpart of relative K-theory (considered in the complex setting in [**4**]) is considered here, some direct image under proper submersion is constructed, and a Grothendieck-Riemann-Roch theorem for Johnson-Nadel-Chern-Simons classes is proved. Metric properties are also studied.

This needs to revisit the construction of η-forms in the case where the direct image is provided by the vertical Euler (de Rham) operator. A direct image under proper submersions of some "non hermitian smooth" or "free multiplicative" K-theory is deduced (in the same context).

Double submersions are also studied to establish some functoriality properties of these direct images.

Résumé **(Image directe pour certaines K-théories secondaires).** — On construit un morphisme d'image directe par submersion propre pour la version réelle de la K-théorie relative (considérée dans [**4**] dans un contexte holomorphe), et un théorème de type Grothendieck-Riemann-Roch est établi pour les classes de Johnson-Nadel-Chern-Simons. On étudie aussi des propriétés métriques.

Ceci nécessite de construire des formes η (de transgression du théorème d'indice des familles) dans le cas où l'image directe est définie par l'operateur d'Euler (de Rham) des fibres. On en déduit également un morphisme d'image directe pour une K-théorie « lisse non hermitienne » ou « multiplicative libre ».

La question de la fonctorialité de ces images directes pour des doubles submersions est également abordée.

1. Introduction

In [**35**], Nadel proposed characteristic classes (also considered by Johnson [**23**], see infra) for triples (E, F, f) where E and F are holomorphic vector bundles on

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some Kähler manifold X, and $f\colon E \xrightarrow{\sim} F$ is a C^∞ vector bundle isomorphism. He conjectured that if X is projective, his classes, which take their values in $H^{(0,\mathrm{odd})}(X)$, were projections of the image by the Abel-Jacobi map of the difference of the Chow group valued Chern classes of E and F. Inspired by **[26]** §6, I developped in **[4]** a notion of relative K-theory which appeared as suitably adapted to describe such triples considered by Nadel. This theory measures the kernel of the forgetful map from the K^0-theory of holomorphic vector bundles on X to the usual topological K^0-theory. As such, if X is projective, any pointed fine moduli space of vector bundles on X naturally maps to this relative K-theory. Moreover, it is rationally isomorphic to the Chow subgroup of homologically trivial cycles.

In **[4]**, Johnson-Nadel classes were extended by considering a suitable projection of the Chern-Simons transgression form associated to compatible connections on E and F. The obtained characteristic class was proved to solve a generalised form of Nadel's conjecture.

I realised very recently that D. Johnson already obtained partial results in this direction: in **[24]** it seems that the same classes as considered by Nadel were defined, and in **[23]** some weaker version (than in **[4]**) of the classes were constructed and a weaker version of the "generalized Nadel conjecture" was proved.

[4] also contains direct images and Grothendieck-Riemann-Roch type results for relative K-theory and its characteristic class, for submersions and immersions of smooth projective varieties.

One of the goals of this article is to study the counterpart of this theory in the context of complex flat vector bundles over some real smooth manifold M. The corresponding relative K-theory was defined by Karoubi **[26]** §6 and studied by Karoubi and Dupont **[17]**. It is here described from objects of the form $(E, \nabla_E, F, \nabla_F, f)$ where f is a smooth vector bundle isomorphism between complex vector bundles E and F endowed with flat connections ∇_E and ∇_F (see Definition 4). If M is compact, the pointed algebraic variety $\mathcal{V}_F$ of flat vector bundle structures on some fixed topological vector bundle on M naturally maps to this relative K-theory.

If $\pi\colon M \longrightarrow B$ is a proper submersion, I construct here (see Definition 26 and Theorem 27) a direct image morphism $\pi_*\colon K^0_{\mathrm{rel}}(M) \longrightarrow K^0_{\mathrm{rel}}(B)$. The main technical problem consists in finding a vector bundle isomorphism (or something equivalent) between representatives of $\pi_! E$ and $\pi_! F$ as virtual flat vector bundles on B in such a way that the direct image becomes natural and functorial.

The counterpart here of Johnson-Nadel classes is simply given by Chern-Simons transgression forms in odd degree de Rham cohomology:

$$\mathcal{N}_{\mathrm{ch}}(E, \nabla_E, F, \nabla_F, f) = \big[\widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)\big] \in H^{\mathrm{odd}}_{dR}(X). \tag{1}$$

Because of its rigidity properties, this Chern-Simons class may essentially detect different connected components of the above algebraic variety $\mathcal{V}_F$, and the class of the determinant line bundle (see §2.3).

A Grothendieck-Riemann-Roch type theorem for $\mathcal{N}_{\mathrm{ch}}$ (Theorem 29) is obtained as a by-product of the constructions performed in pursuing the second goal of the article, namely the study of "free multiplicative" or "non hermitian smooth" K-theory. This K-theory, denoted by $\widehat{K}_{\mathrm{ch}}$ is generated by triples of the form (E, ∇, α) where ∇ is a connection on the complex vector bundle E over M and α is an odd degree differential form defined modulo exact forms. Relations are direct sum and if $f\colon E \longrightarrow F$ is any smooth vector bundle isomorphism:

$$(E, \nabla_E, \alpha) = \big(F, \nabla_F, \alpha + \widetilde{\mathrm{ch}}(\nabla_E, f^*\nabla_F)\big) \tag{2}$$

(Here $\widetilde{\mathrm{ch}}$ is again a Chern-Simons transgression form). K^0_{rel} and $\widehat{K}_{\mathrm{ch}}$ are related by a commutative diagram whose lines are exact sequences (see Proposition 10):

$$\begin{array}{ccccccc}
K^1_{\mathrm{top}}(M) & \longrightarrow & K^0_{\mathrm{rel}}(M) & \longrightarrow & K^0_{\mathrm{flat}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M) \\
\downarrow \| & & \downarrow \mathcal{N}_{\mathrm{ch}} & & \downarrow & & \downarrow \| \\
K^1_{\mathrm{top}}(M) & \xrightarrow{\mathrm{ch}} & \Omega^{\mathrm{odd}}(M)/d\Omega^{\mathrm{even}}(M) & \longrightarrow & \widehat{K}_{\mathrm{ch}}(M) & \longrightarrow & K^0_{\mathrm{top}}(M)
\end{array} \tag{3}$$

In this diagram, $\Omega^\bullet(M)$ denotes differential forms, K_{top} denotes ordinary K-theory, and K^0_{flat} denotes the K^0 theory of the category of flat bundles modulo exact sequences. For any vector bundle E on M endowed with a flat connection ∇_E, the image in $\widehat{K}_{\mathrm{ch}}(M)$ of $(E, \nabla_E) \in K^0_{\mathrm{flat}}(M)$ is the triple $(E, \nabla_E, 0)$.

On one hand, Karoubi's multiplicative K-theory **[26]** **[27]** **[28]** consists of quotients (the form α being defined modulo greater subgroups than only exact forms) of subgroups (defined by restrictions on the Chern-Weil character form $\mathrm{ch}(\nabla)$) of this theory. These subgroups and constraints stemm from natural filtrations of the de Rham complex of M suitably adapted to the geometry studied. In **[28]**, Karoubi studies foliations for which he constructs generalisations of the Godbillon-Vey invariant, and holomorphic and algebraic varieties for which known characteristic classes for holomorphic or algebraic vector bundles are shown to factor through the suitable multiplicative K-theory. Poutriquet **[36]** studies the context of conical singularities. The corresponding multiplicative K-theory he constructs shows interesting similarities with intersection cohomology. Felisatti and Neumann **[18]** generalise the concept of multiplicative K-theory to simplicial manifolds with applications to classifying spaces of Lie groups and Lie groupoids.

As an example, the multiplicative K-theory adapted to the study of flat bundles is the subgroup of $\widehat{K}_{\mathrm{ch}}$ generated by triples (E, ∇, α) such that

$$\mathrm{ch}(\nabla) - d\alpha \in \mathbb{Z} \subset \Omega^{\mathrm{even}}(M) \tag{4}$$

Removing this constraint would justify the name "free multiplicative" K-theory. Direct image results for $\widehat{K}_{\mathrm{ch}}$ should have corollaries for "nonfree" multiplicative K-theories under mild compatibility conditions on the filtrations of the de Rham complex used to define them.

On the other hand, Bunke and Schick [**14**] defined a smooth (hermitian) K-theory, which coincides with the subgroup of $\widehat{K}_{\mathrm{ch}}$ generated by triples (E, ∇, α) where α is a real form and ∇ respects some hermitian metric on E. Bunke and Schick's smooth K-theory is motivated by quantum field theory considerations [**19**] and it fits in the general framework of smooth extensions of generalized cohomology theories [**20**] [**21**]. Among other examples, Bunke and Schick construct interesting smooth K-theory canonical classes on homogeneous spaces and generalisations of parametrized ρ-invariants [**14**] §5.

Allowing nonunitary connections (and nonreal forms) would justify the name "non hermitian smooth K-theory". Anyway, the hermitian restriction would prevent from obtaining a natural morphism $K^0_{\mathrm{flat}}(M) \longrightarrow \widehat{K}_{\mathrm{ch}}(M)$ because of the existence of nonunitary flat vector bundles.

The obstruction for a flat bundle (E, ∇_E) to be unitary can be detected by characteristic classes similar to $\mathscr{N}_{\mathrm{ch}}(E, \nabla_E, E, \nabla_E^*, \mathrm{Id}_E)$ where ∇_E^* is the adjoint connection of ∇_E with respect to any hermitian metric on E (22). Such classes were first considered by Kamber and Tondeur [**25**], they correspond to the imaginary part of Chern-Cheeger-Simons classes [**15**], (see [**11**] Proposition 1.14). Karoubi proved in [**26**] §6.31 that they could detect some Borel generators of algebraic K-theory of integer rings in number fields [**12**]. See also [**11**] §I(g) for an interpretation as stable characteristic classes arising from stable continuous cohomology of $GL(\mathbb{C})$.

Here this Borel-Kamber-Tondeur class is extended to $\widehat{K}_{\mathrm{ch}}$. It is not always a cohomology class, but rather a purely imaginary differential form defined modulo exact forms (see Definition 16).

Moreover, a direct image morphism for $\widehat{K}_{\mathrm{ch}}$ under proper submersions is constructed (Theorem 31), which is compatible with the usual (sheaf theoretic) direct image of flat vector bundles (using fiberwise twisted de Rham cohomology, see Definition 22). This is performed from the families analytic index of the fiberwise twisted Euler operator together with a suitable η-form which is a non hermitian generalisation of that of Bunke [**13**] (Theorem 28). Functoriality is established only for the "nonfree" multiplicative subgroup of $\widehat{K}_{\mathrm{ch}}$ subject to the constraint (4), using some universal characterisation of the η-form.

Finally the symmetries induced by the fiberwise Hodge star operator are studied. Reality (resp. vanishing) properties of the pushforwards are established in the even (resp. odd) dimensional fibre case (Theorems 32 and 33).

The paper is organized as follows: the definitions of K-theories and characteristic classes, and their mutual relations are given in §2, the pushforward morphisms are defined and all the theorems are stated in §3, the construction of the direct image for relative K-theory is performed in §4, the construction of the η-form and all its consequences are detailed in §5, and §6 is devoted to results about symmetries induced by the fiberwise Hodge star operator. Finally, double fibrations are studied in §7. This paper is a reformulation of previously diffused preprints. I apologize for some changes of title, names and notations between earlier versions and this one.

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2. Various K-theories

After recalling some facts about Chern-Simons transgression in §2.1, the definitions of all the K-theory groups considered here are given in §2.2. §2.3 is devoted to the counterpart of Johnson-Nadel's classes defined in [**4**], §2.4 to the diagrams and exact sequences in which these K-groups enter, §2.5 and §2.6 to hermitian metrics and the extended Borel-Kamber-Tondeur class on $\widehat{K}_{\mathrm{ch}}$.

2.1. Preliminaries

2.1.1. *Connections and vector bundle morphisms.* — Let M be a smooth manifold. Let E and F be two vector bundles on M. Two vector bundles isomorphisms f and $g\colon E \xrightarrow{\sim} F$ are called isotopic if there exists a smooth family $(f_t)_{t\in[0,1]}$ of isomorphisms $f_t\colon E \xrightarrow{\sim} F$ such that $f_0 = f$ and $f_1 = g$. Suppose that E and F are endowed with connections ∇_E and ∇_F respectively (which need not be flat). A vector bundle morphism (which does not need to be an isomorphism) $f\colon E \longrightarrow F$ is parallel if $\nabla_F \circ f = f \circ \nabla_E$. For three vector bundles E', E and E'' endowed with connections $\nabla_{E'}$, ∇_E and $\nabla_{E''}$, the short exact sequence

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{p} E'' \longrightarrow 0 \tag{5}$$

is parallel if the morphisms i and p are parallel with respect to $\nabla_{E'}$, ∇_E and $\nabla_{E''}$. Parallel longer exact sequences or complexes of vector bundles are defined in a similar obvious way. In such parallel long exact sequences (or complexes), the kernel or image subbundles are respected by the connections of their ambient bundles (which are not supposed to be flat), so that cokernel or coimage bundles inherit natural connections (which need not be flat). Thus, longer parallel exact sequences (or complexes) can be