*Astérisque*
**327**, 2009, p. 361–424

# HERMITIAN VECTOR BUNDLES AND EXTENSION GROUPS ON ARITHMETIC SCHEMES II. THE ARITHMETIC ATIYAH EXTENSION

*by*

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***Abstract.*** — In a previous paper, we have defined arithmetic extension groups in the context of Arakelov geometry. In the present one, we introduce an arithmetic analogue of the Atiyah extension that defines an element — the arithmetic Atiyah class — in a suitable arithmetic extension group. Namely, if $\overline{E}$ is a hermitian vector bundle on an arithmetic scheme $X$, its arithmetic Atiyah class $\widehat{\mathrm{at}}_{X/\mathbb{Z}}(\overline{E})$ lies in the group $\widehat{\mathrm{Ext}}^1_X(E, E \otimes \Omega^1_{X/\mathbb{Z}})$, and is an obstruction to the algebraicity over $X$ of the unitary connection on the vector bundle $E_{\mathbb{C}}$ over the complex manifold $X(\mathbb{C})$ that is compatible with its holomorphic structure.

In the first sections of this article, we develop the basic properties of the arithmetic Atiyah class which can be used to define characteristic classes in arithmetic Hodge cohomology.

Then we study the vanishing of the first Chern class $\hat{c}_1^H(\overline{L})$ of a hermitian line bundle $\overline{L}$ in the arithmetic Hodge cohomology group $\widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/\mathbb{Z}})$. This may be translated into a concrete problem of diophantine geometry, concerning rational points of the universal vector extension of the Picard variety of $X$. We investigate this problem, which was already considered and solved in some cases by Bertrand, by using a classical transcendence result of Schneider-Lang, and we derive a finiteness result for the kernel of $\hat{c}_1^H$.

In the final section, we consider a geometric analog of our arithmetic situation, namely a smooth, projective variety $X$ which is fibered on a curve $C$ defined over some field $k$ of characteristic zero. To any line bundle $L$ over $X$ is attached its relative Atiyah class $\mathrm{at}_{X/C}L$ in $H^1(X, \Omega^1_{X/C})$. We describe precisely when $\mathrm{at}_{X/C}L$ vanishes. In particular, when the fixed part of the relative Picard variety of $X$ over $C$ is trivial, this holds iff some positive power of $L$ descends to a line bundle over $C$.

***2010 Mathematics Subject Classification.*** — 14G40; 11J95, 14F05, 32L10.
***Key words and phrases.*** — Arakelov geometry, hermitian vector bundles, extension groups, Atiyah class, transcendence and algebraic groups.

***Résumé*** **(Fibrés vectoriels hermitiens et groupes d'extensions sur les schémas arithmétiques II. La classe d'Atiyah arithmétique)**

Dans un précédent article, nous avons défini des groupes d'extensions arithmétiques dans le contexte de la géométrie d'Arakelov. Dans le présent travail, nous introduisons un analogue arithmétique de l'extension d'Atiyah; sa classe dans un groupe d'extensions arithmétiques convenable définit la classe d'Atiyah arithmétique. Plus précisément, pour tout fibré vectoriel hermitien $\overline{E}$ sur un schéma arithmétique $X$, sa classe d'Atiyah arithmétique $\widehat{\mathrm{at}}_{X/\mathbb{Z}}(\overline{E})$ appartient au groupe $\widehat{\mathrm{Ext}}^1_X(E, E \otimes \Omega^1_{X/\mathbb{Z}})$ et constitue une obstruction à l'algébricité sur $X$ de l'unique connection unitaire sur la fibré vectoriel $E_{\mathbb{C}}$ sur la variété complexe $X(\mathbb{C})$ qui soit compatible avec sa structure holomorphe.

Dans les premières sections de cet article, nous présentons la construction et les propriétés de base de la classe d'Atiyah, qui permettent notamment de définir des classes caractéristiques en cohomologie de Hodge arithmétique.

Nous étudions ensuite l'annulation de la première classe de Chern $\hat{c}_1^H(\overline{L})$ d'un fibré en droites hermitien $\overline{L}$ dans le groupe de cohomologie de Hodge arithmétique $\widehat{\mathrm{Ext}}^1_X(\mathcal{O}_X, \Omega^1_{X/\mathbb{Z}})$. La détermination de tels fibrés en droites hermitiens se traduit en une question de géométrie diophantienne, concernant les points rationnels de l'extension vectorielle universelle de la variété de Picard de $X$. Nous étudions ce problème — qui a déjà été considéré, et résolu dans certains cas, par Bertrand — au moyen d'un classique résultat de transcendance dû à Schneider et Lang, et nous en déduisons un théorème de finitude sur le noyau de $\hat{c}_1^H$.

Dans la dernière section, nous étudions un analogue géométrique de la situation arithmétique précédente. A savoir, nous considérons une variété projective lisse $X$ fibrée sur une courbe $C$, au dessus d'un corps de base $k$ de caractéristique nulle et nous attachons à tout fibré en droites $L$ sur $X$ sa classe d'Atiyah relative $\mathrm{at}_{X/C}L$ dans $H^1(X, \Omega^1_{X/C})$. Nous déterminons quand cette classe $\mathrm{at}_{X/C}L$ s'annule. Notamment, lorsque la variété de Picard relative de $X$ sur $C$ n'a pas de partie fixe, cela se produit précisément lorsque une puissance non-nulle de $L$ descend en un fibré en droites sur $C$.

## 0. Introduction

**0.1.** — This paper is a sequel to [**7**], where we have defined and investigated arithmetic extensions on arithmetic schemes, and the groups they define.

Recall that if $X$ is a scheme over $\operatorname{Spec}\mathbb{Z}$, separated of finite type, whose generic fiber $X_{\mathbb{Q}}$ is smooth, then an arithmetic extension of vector bundles over $X$ is the data $(\mathcal{E}, s)$ of a short exact sequence of vector bundles (that is, of locally free coherent sheaves of $\mathcal{O}_X$-modules) on the scheme $X$,

$$\mathcal{E} : 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0, \tag{0.1}$$

and of a $\mathcal{C}^\infty$-splitting

$$s : F_{\mathbb{C}} \longrightarrow E_{\mathbb{C}},$$

invariant under complex conjugation, of the extension of $\mathcal{C}^\infty$-complex vector bundles on the complex manifold $X(\mathbb{C})$

$$\mathcal{E}_{\mathbb{C}} : 0 \longrightarrow G_{\mathbb{C}} \xrightarrow{i_{\mathbb{C}}} E_{\mathbb{C}} \xrightarrow{p_{\mathbb{C}}} F_{\mathbb{C}} \longrightarrow 0$$

that is deduced from $\mathcal{E}$ by the base change from $\mathbb{Z}$ to $\mathbb{C}$ and analytification.

For any two given vector bundles $F$ and $G$ over $X$, the isomorphism classes of the so-defined arithmetic extensions of $F$ by $G$ constitute a set $\widehat{\mathrm{Ext}}^1_X(F,G)$ that becomes an abelian group when equipped with the addition law defined by a variant of the classical construction of the Baer sum of 1-extensions of (sheaves of) modules[(1)].

Recall that a hermitian vector bundle $\overline{E}$ over $X$ is a pair $(E, \|.\|)$ consisting of a vector bundle $E$ over $X$ and of a $\mathcal{C}^\infty$-hermitian metric, invariant under complex conjugation, on the holomorphic vector bundle $E_{\mathbb{C}}$ over $X(\mathbb{C})$. Examples of arithmetic extensions in the above sense are provided by admissible extensions

$$\overline{\mathcal{E}} : 0 \longrightarrow \overline{G} \xrightarrow{i} \overline{E} \xrightarrow{p} \overline{F} \longrightarrow 0 \tag{0.2}$$

of hermitian vector bundles over $X$, namely from the data of an extension

$$\mathcal{E} : 0 \longrightarrow G \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0$$

of the underlying $\mathcal{O}_X$-modules such that the hermitian metrics $\|.\|_{\overline{G}}$ and $\|.\|_{\overline{F}}$ on $G_{\mathbb{C}}$ and $F_{\mathbb{C}}$ are induced (by restriction and quotients) by the metric $\|.\|_{\overline{E}}$ on $E_{\mathbb{C}}$ (by means of the morphisms $i_{\mathbb{C}}$ and $p_{\mathbb{C}}$). Indeed, to any such admissible extension is naturally attached its orthogonal splitting, namely the $\mathcal{C}^\infty$-splitting

$$s_{\overline{\mathcal{E}}} : F_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$$

that maps $F_{\mathbb{C}}$ isomorphically onto the orthogonal complement $i_{\mathbb{C}}(G_{\mathbb{C}})^\perp$ of the image of $i_{\mathbb{C}}$ in $E_{\mathbb{C}}$. This splitting is invariant under complex conjugation, and $(\mathcal{E}, s_{\overline{\mathcal{E}}})$ is an arithmetic extension of $F$ by $G$. For any two hermitian vector bundles $\overline{F}$ and $\overline{G}$ over $X$, this construction establishes a bijection from the set of isomorphism classes of admissible extension of the form (0.2) to the set $\widehat{\mathrm{Ext}}^1_X(F,G)$.

In **[7]** we studied basic properties of the so-defined arithmetic extension groups. In particular, we introduced the following natural morphisms of abelian groups:

– the "forgetful" morphism

$$\nu : \widehat{\mathrm{Ext}}^1_X(F,G) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(F,G),$$

which maps the class of an arithmetic extension $(\mathcal{E}, s)$ to the one of the underlying extension $\mathcal{E}$ of $\mathcal{O}_X$-modules;

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(1) Consider indeed two arithmetic extensions of $F$ by $G$, say $\overline{\mathcal{E}}_\alpha := (\mathcal{E}_\alpha, s_\alpha)$, $\alpha = 1, 2$, defined by extensions of vector bundles $\mathcal{E}_\alpha : 0 \to G \xrightarrow{i_\alpha} E_\alpha \xrightarrow{p_\alpha} F \to 0$ and $\mathcal{C}^\infty$-splittings $s_\alpha : F_{\mathbb{C}} \to E_{\alpha,\mathbb{C}}$. We may define a vector bundle $E := \frac{\mathrm{Ker}(p_1 - p_2 : E_1 \oplus E_2 \to F)}{\mathrm{Im}\,((i_1, -i_2) : G \to E_1 \oplus E_2)}$ over $X$. The Baer sum of $\overline{\mathcal{E}}_1$ and $\overline{\mathcal{E}}_2$ is the arithmetic extension $\overline{\mathcal{E}}$ defined by the usual Baer sum of $\mathcal{E}_1$ and $\mathcal{E}_2$ — namely $\mathcal{E} : 0 \to G \xrightarrow{i} E \xrightarrow{p} F \to 0$ where the morphisms $i : G \to E$ and $p : E \to F$ are defined by $p([(g_1, g_2)]) := p_1(f_1) = p_2(f_2)$ and $i(g) := [(i_1(g), 0)] = [(0, i_2(g))]$ — equipped with the $\mathcal{C}^\infty$-splitting $s : F_{\mathbb{C}} \to E_{\mathbb{C}}$ defined by $s(e) := [(s_1(e), s_2(e))]$.

– the morphism

$$b : \mathrm{Hom}_{\mathscr{C}^\infty_{X(\mathbb{C})}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty} \longrightarrow \widehat{\mathrm{Ext}}^1_X(F, G),$$

defined on the real vector space $\mathrm{Hom}_{\mathscr{C}^\infty_{X(\mathbb{C})}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty}$ of $\mathscr{C}^\infty$-morphisms of vector bundles over $X(\mathbb{C})$ from $F_{\mathbb{C}}$ to $G_{\mathbb{C}}$, invariant under complex conjugation; it sends an element $T$ in this space to the class of the arithmetic extension $(\mathscr{E}, s)$ where $\mathscr{E}$ is the trivial algebraic extension, defined by (0.1) with $E := G \oplus F$ and $i$ and $p$ the obvious injection and projection morphisms, and where $s$ is given by $s(f) = (T(f), f)$;

– the morphism

$$\iota : \mathrm{Hom}_{\mathscr{O}_X}(F, G) \longrightarrow \mathrm{Hom}_{\mathscr{C}^\infty_{X(\mathbb{C})}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty}$$

which sends a morphism $\varphi : F \to G$ of vector bundles over $X$ to the morphism of $\mathscr{C}^\infty$-complex vector bundles $\varphi_{\mathbb{C}} : F_{\mathbb{C}} \to G_{\mathbb{C}}$ deduced from $\varphi$ by base change from $\mathbb{Z}$ to $\mathbb{C}$ and analytification;

– the morphism

$$\Psi : \widehat{\mathrm{Ext}}^1_X(F, G) \longrightarrow Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G),$$

that takes values in the real vector space

$$Z^{0,1}_{\overline{\partial}}(X_{\mathbb{R}}, F^\vee \otimes G) := Z^{0,1}_{\overline{\partial}}(X(\mathbb{C}), F^\vee_{\mathbb{C}} \otimes G_{\mathbb{C}})^{F_\infty}$$

of $\overline{\partial}$-closed forms of type $(0,1)$ on $X(\mathbb{C})$ with coefficients in $F^\vee_{\mathbb{C}} \otimes G_{\mathbb{C}}$, invariant under complex conjugation. It maps the class of an arithmetic extension $(\mathscr{E}, s)$ to its "second fundamental form" $\Psi(\mathscr{E}, s)$ defined by

$$i_{\mathbb{C}} \circ \Psi(\mathscr{E}, s) = \overline{\partial}_{F^\vee_{\mathbb{C}} \otimes G_{\mathbb{C}}}(s).$$

We also established the following basic exact sequence:

$$\mathrm{Hom}_{\mathscr{O}_X}(F, G) \xrightarrow{\iota} \mathrm{Hom}_{\mathscr{C}^\infty_{X(\mathbb{C})}}(F_{\mathbb{C}}, G_{\mathbb{C}})^{F_\infty} \xrightarrow{b} \widehat{\mathrm{Ext}}^1_X(F, G) \xrightarrow{\nu} \mathrm{Ext}^1_{\mathscr{O}_X}(F, G) \to 0, \tag{0.3}$$

which displays the arithmetic extension group $\widehat{\mathrm{Ext}}^1_X(F, G)$ as an extension of the "classical" extension group $\mathrm{Ext}^1_{\mathscr{O}_X}(F, G)$ by a group of analytic type.

The sequel of **[7]** was devoted to the study of the groups $\widehat{\mathrm{Ext}}^1_X(F, G)$ when the base scheme is an arithmetic curve, that is, the spectrum Spec $\mathscr{O}_K$ of the ring of integers of some number field $K$. In this special case, these extension groups appear as natural tools in geometry of numbers and reduction theory in their modern guise, namely the study of hermitian vector bundles over arithmetic curves and their admissible extensions.

In the present paper, we focus on a natural construction of arithmetic extensions attached to hermitian vector bundles over an arithmetic scheme $X$ as above, their *arithmetic Atiyah extensions.* In contrast with the arithmetic extensions over arithmetic curves investigated in [**7**], for which the interpretation as admissible extensions was crucial, the arithmetic Atiyah extensions are genuine examples of arithmetic extensions constructed as pairs $(\mathcal{E}, s)$ — where $s$ is a $\mathcal{C}^\infty$-splitting of an extension $\mathcal{E}$ of vector bundles over $X$ — and not derived from an admissible extension.

**0.2.** — Atiyah extensions of vector bundles were initially introduced by Atiyah [**2**] in the context of complex analytic geometry.

Namely, for any holomorphic vector bundle $E$ over a complex manifold $X$, Atiyah introduces the holomorphic vector bundle $P^1_X(E)$ of jets of order one of sections of $E$, whose fiber $P^1_X(E)_x$ at a point $x$ of $X$ is by definition the space of sections of $E$ over the first order thickening $x_1 := \operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{m}_x^2$ of $x$ in $X$. Here, as usual, $\mathcal{O}_X$ denotes the sheaf of holomorphic functions over $X$, and $\mathfrak{m}_x$ the maximal ideal of its stalk $\mathcal{O}_{X,x}$ at $x$.

The vector bundle $P^1_X(E)$ fits into a short exact sequence of holomorphic vector bundles

$$\mathcal{A}t_X E : 0 \longrightarrow E \otimes \Omega^1_X \xrightarrow{i} P^1_X(E) \xrightarrow{p} E \longrightarrow 0, \tag{0.4}$$

where the morphisms $i$ and $p$ are defined as follows: for any point $x$ in $X$, the map $i_x : E_x \otimes \Omega^1_{X,x} \to P^1_X(E)_x$ maps an element $v$ in $E_x \otimes \Omega^1_{X,x} \simeq \operatorname{Hom}_{\mathbb{C}}(T_{X,x}, E_x)$ to the section of $E$ over $x_1$ that vanishes at $x$ and admits $v$ as differential, while the map $p_x : P^1_X(E)_x \to E_x$ is simply the evaluation at $x$.

The Atiyah extension of $E$ is precisely the extension $\mathcal{A}t_X E$ of $E$ by $E \otimes \Omega^1_X$ so-defined. According to its very definition, its class $\mathrm{at}_X E$ in the group $\operatorname{Ext}^1_{\mathcal{O}_X}(E, E \otimes \Omega^1_X)$ which classifies extensions of holomorphic vector bundles of $E$ by $E \otimes \Omega^1_X$ is the obstruction to the existence of a holomorphic connection

$$\nabla : E \longrightarrow E \otimes \Omega^1_X$$

on the vector bundle $E$.

The point of Atiyah's article [**2**] is that the class $\mathrm{at}_X E$ also leads to a straightforward construction of characteristic classes of $E$ with values in the so-called Hodge cohomology groups of $X$

$$H^{p,p}(X) := H^p(X, \Omega^p_X). \tag{0.5}$$

For instance, Atiyah defines a first Chern class $c_1^H(E)$ in $H^{1,1}(X) = H^1(X, \Omega^1_X)$ as the image of $\mathrm{at}_X E$ by the morphism

$$\begin{array}{ccccc} \operatorname{Ext}^1_{\mathcal{O}_X}(E, E \otimes \Omega^1_X) & \simeq & \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}nd\, E \otimes \Omega^1_X) & & \\ & & \downarrow {\scriptstyle (\mathrm{Tr}_E \otimes \mathrm{id}_{\Omega^1_X}) \circ \_} & & \\ & & \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^1_X) & \simeq & H^1(X, \Omega^1_X) \end{array}$$