

441

ASTÉRISQUE

2023

BROWNIAN STRUCTURE IN THE KPZ FIXED POINT

Jacob CALVERT, Alan HAMMOND & Milind HEGDE

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Astérisque est un périodique de la Société Mathématique de France.

Numéro 441, 2023

Comité de rédaction

Marie-Claude ARNAUD	Alexandru OANCEA
Christophe BREUIL	Nicolas RESSAYRE
Philippe EYSSIDIEUX	Rémi RHODES
Colin GUILLARMOU	Sylvia SERFATY
Fanny KASSEL	Sug Woo SHIN
Eric MOULINES	

Nicolas BURQ (dir.)

Diffusion

Maison de la SMF	AMS
Case 916 - Luminy	P.O. Box 6248
13288 Marseille Cedex 9	Providence RI 02940
France	USA

commandes@smf.emath.fr <http://www.ams.org>

Tarifs

Vente au numéro : 38 € (\$57)
Abonnement Europe : 665 €, hors Europe : 718 € (\$1077)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat

Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Fax: (33) 01 40 46 90 96
asterisque@smf.emath.fr • <http://smf.emath.fr/>

© Société Mathématique de France 2023

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN: 0303-1179 (print) 2492-5926 (electronic)
ISBN 978-2-85629-973-9
doi:10.24033/ast.1200

Directeur de la publication : Fabien Durand

441

ASTÉRISQUE

2023

BROWNIAN STRUCTURE IN THE KPZ FIXED POINT

Jacob CALVERT, Alan HAMMOND & Milind HEGDE

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Jacob Calvert

Department of Statistics, U.C. Berkeley, 451 Evans Hall, Berkeley, CA, 94720-3840,
U.S.A

`jacob_calvert@berkeley.edu`

Alan Hammond

Departments of Mathematics and Statistics, U.C. Berkeley, 899 Evans Hall, Berkeley,
CA, 94720-3840, U.S.A

`alanmh@berkeley.edu`

Milind Hegde

Department of Mathematics, Columbia University, 621 Mathematics Hall, New York,
NY, 10027, U.S.A

`milind.hegde@columbia.edu`

Texte reçu le 24 décembre 2019, révisé le 5 mars 2022, accepté le 27 septembre 2022.

Mathematical Subject Classification (2010). — 82C22, 82B23, 60H15.

Keywords. — Brownian last passage percolation, multi-line Airy process, Airy line ensemble.

Mots-clefs. — Percolation de passage du dernier brownien, processus multi-ligne d'Airy, ensemble de lignes d'Airy.

BROWNIAN STRUCTURE IN THE KPZ FIXED POINT

by Jacob CALVERT, Alan HAMMOND & Milind HEGDE

Abstract. — Many models of one-dimensional local random growth are expected to lie in the Kardar-Parisi-Zhang (KPZ) universality class. For such a model, the interface profile at advanced time may be viewed in scaled coordinates specified via characteristic KPZ scaling exponents of one-third and two-thirds. When the long time limit of this scaled interface is taken, it is expected—and proved for a few integrable models—that, up to a parabolic shift, the Airy_2 process $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}$ is obtained. This process may be embedded via the Robinson-Schensted-Knuth correspondence as the uppermost curve in an \mathbb{N} -indexed system of random continuous curves, the Airy line ensemble.

Among our principal results is the assertion that the Airy_2 process enjoys a very strong similarity to Brownian motion B (of rate two) on unit-order intervals. This result yields bounds on the Airy_2 probabilities of a large class of events from the counterpart bounds on Brownian motion probabilities. The result has the consequence that the Radon-Nikodym derivative of the law of \mathcal{A} on say $[-1, 1]$ after a suitable vertical shift, with respect to the law of B on the same interval, lies in every L^p space for $p \in (1, \infty)$. In fact, the quantitative comparison of probability bounds we prove also holds for the scaled energy profile with Dirac delta initial condition of the model of Brownian last passage percolation, a model that lies in the KPZ universality class and in which the energy of paths in a random Brownian environment is maximized.

Our technique of proof harnesses a probabilistic resampling or *Brownian Gibbs* property satisfied by the Airy line ensemble after parabolic shift, and this article develops Brownian Gibbs analysis of this ensemble begun in work of Corwin and Hammond (2014) and pursued by Hammond (2019). Our Brownian comparison for scaled interface profiles is an element in the ongoing program of studying KPZ universality via probabilistic and geometric methods of proof, aided by limited but essential use of integrable inputs. Indeed, the comparison result is a useful tool for studying this universality class. We present and prove several applications, concerning for example the structure of near ground states in Brownian last passage percolation, or Brownian structure in scaled interface profiles that arise from the evolution from any element in a very general class of initial data.

Résumé. (Structure brownienne dans le point fixe KPZ) — On s’attend à ce que de nombreux modèles de croissance aléatoire uni-dimensionnelle appartiennent à la classe d’universalité de Kardar-Parisi-Zhang (KPZ). En changeant convenablement le temps d’échelle via les exposants caractéristiques de KPZ d’un tiers et de deux tiers, on considère le profil déterminé par un tel modèle. Dans la limite de temps infini, ce profil devrait converger vers le processus $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}$ à l’addition d’une parabole près; constatation dont la preuve est connue dans les cas de certains modèles intégrables. Ce processus peut-être identifié par la correspondance de Robinson-Schensted-Knuth à la courbe la plus haute d’un système indexé par \mathbb{N} de courbes aléatoires continues: l’ensemble de lignes d’Airy. Parmi nos résultats principaux est l’énoncé que le processus Airy_2 présente une grande similarité avec le mouvement brownien (de taux deux) sur un intervalle de longueur de l’ordre de l’unité. Grâce à ce résultat, nous obtenons des bornes sur les probabilités Airy_2 d’une large classe d’événements à partir de bornes browniennes homologues. Ce résultat a le corollaire que la dérivée Radon-Nikodym de la loi $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}$ sur l’intervalle $[-1, 1]$ par rapport à la loi brownienne sur cet intervalle appartient à tous les espaces L^p où $p \in (1, \infty)$. En fait, la comparaison quantitative de bornes probabilistiques est également vraie pour le profil d’énergie après changement d’échelle (avec condition initiale delta Dirac) du modèle de percolation de passage dernier brownien, un modèle qui appartient à la classe d’universalité de KPZ et dans lequel l’énergie de chemins est maximisée dans un environnement aléatoire brownien. Notre technique de preuve utilise une propriété de rééchantillonnage probabiliste, la propriété de Gibbs brownienne, qui est satisfaite par l’ensemble parabolique de lignes d’Airy; cet article développe l’analyse de Gibbs brownienne de cet ensemble commencée par Corwin et Hammond (2014) et poursuivie par Hammond (2019). Notre comparaison brownienne de profils après changement d’échelle est un élément du programme plus vaste de l’étude de l’universalité de KPZ par les moyens géométriques et probabilistiques, facilité par l’usage, limité mais critique, des énoncés intégrables. En effet, le résultat de comparaison est un outil puissant dans l’étude de cette classe d’universalité. Nous prouvons quelques corollaires au sujet de la structure d’états quasi-minimaux dans la percolation de dernier passage brownien, ou de la structure brownienne trouvée dans les profils d’interface après changement d’échelle donnée par l’évolution d’un élément arbitraire dans une classe très large de conditions initiales.

CONTENTS

1. Introduction	1
1.1. KPZ universality	1
1.2. Applications of Theorem 1.1	7
1.3. Pertinent recent work	11
1.4. A form of Brownian regularity for weight profiles with general initial conditions	16
1.5. Method of proof	19
1.6. Organization of the article	21
2. Proofs of applications	23
2.1. Local Johansson	23
2.2. Many near touch	25
3. Notation and setup	29
3.1. Notation, Brownian Gibbs, and regular ensembles	29
3.2. An important example of regular ensembles: Brownian LPP weight profiles	32
3.3. Main result	35
4. Proof framework	37
4.1. The jump ensemble	37
4.2. A conceptual framework in terms of costs	57
5. Proving the density bounds	71
5.1. The easy case: Below the pole on both sides	71
5.2. The moderate case: Above the pole on both sides	73
5.3. The difficult case: Above and below the pole on either side	84
5.4. When no pole is present	91
6. The patchwork quilt resewn	95
6.1. Brownian motion regularity of weight profiles with general initial conditions	95
6.2. The limiting patchwork quilt	97
A. Brownian meander calculations	105
B. Measurability of the jump ensemble	111
Bibliography	115

CHAPTER 1

INTRODUCTION

1.1. KPZ universality

The field of Kardar-Parisi-Zhang (KPZ) universality concerns one-dimensional interfaces that evolve randomly in time, and universal random structures that describe, independently of the microscopic details that specify the local evolution of such random models, the geometry and fluctuation of the interface when time is advanced. The KPZ universality class, whose members are random processes that are expected to evince these late-time characteristics, is very broad. The basic features of a random growth model that indicate that it may be a member of the KPZ universality class are that growth occurs in a direction normal to the present local slope of the interface at a rate influenced by the slope, alongside two competing forces: a smoothing effect generated by surface tension, and a roughening effect caused by forces in the environment that are local and random.

The principal results of this monograph offer a very strong assertion concerning the geometry of the canonical random object—the *Airy₂ process with parabolic curvature*—that describes the scaled attributes of what are known as the narrow wedge solutions to models in the KPZ universality class. Our results further offer a counterpart description that holds in a rather uniform sense in the prelimit for a model in the KPZ class known as *Brownian last passage percolation* (LPP). The geometric inference that we make is a powerful assertion of the Brownian nature of the scaled interface profile and it is the engine for a wide array of applications about scaled random growth models and their KPZ universality limiting structure.

The parabolically shifted Airy₂ process is the first in a family of limiting processes, indexed by the initial condition, which are expected to be universal objects in the KPZ universality class, and which may be referred to collectively as the *KPZ fixed point*. The name arises from the physical view that such objects are expected to be fixed points of suitable renormalization operators. (Indeed, an important related object has been constructed by [38] in a paper bearing the name “The KPZ fixed point”. Our usage of this term is a corruption of theirs, but only mildly so. We will comment further on [38] in Section 1.3.2.) One application of our results will be a statement

about a certain form of Brownian regularity for these KPZ fixed point profiles, i.e., limiting scaled interface profiles from general initial data in Brownian LPP.

We defer the definition of Brownian last passage percolation and the presentation of our main theorem, expressed in prelimiting terms that capture Brownian LPP, to Section 3.2. What we are able to indicate in the ensuing paragraphs is the form of our principal assertion in the limiting case of scaled KPZ structure. Indeed, we will next introduce the parabolic Airy₂ process; and then we state our principal conclusion as it applies to this process.

The Airy₂ process $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}$ is a stationary process first introduced by Prähofer and Spohn [47] in a scaled description of the polynuclear growth (PNG) model; or equivalently, of another famous last passage percolation model, Poissonian LPP. It was defined in [47] in terms of its finite-dimensional distributions—written via determinantal formulas involving the Airy kernel—and shown to have a continuous version. Its basic role in KPZ universality may be expressed via the scaled “narrow wedge” interface $L_n : \mathbb{R} \rightarrow \mathbb{R}$ for various KPZ models, where n is an abstract parameter measuring the size of the system in the model. It is a major open problem to rigorously show in a broad class of models that the distributional high- n limit, with respect to the topology of locally uniform convergence on the space of continuous functions with domain and co-domain the real line, of the process $x \mapsto L_n(x)$ (after adopting model-dependent constants in the scalings) is $\mathcal{A}(x) - x^2$, a parabolically shifted Airy₂ process.

This inference has recently been proven for the KPZ equation [50, 55], and had been validated for several stochastic growth models with narrow wedge initial data beyond PNG. These models include the totally asymmetric simple exclusion process (TASEP) [8] as well as the model that will be the principal object of rigorous attention in this monograph, namely Brownian LPP. In the latter case, the convergence is proved via a distributional relation with Dyson Brownian motion that will be reviewed in Section 3.2.

1.1.1. Locally Brownian nature of the limiting process. — In the case of Brownian LPP, as mentioned, the limiting process is $\mathcal{A}(x) - x^2$. We define

$$\mathcal{L}(x) := 2^{-1/2} (\mathcal{A}(x) - x^2),$$

and call it the *parabolic Airy₂ process*, in spite of the factor of $2^{-1/2}$. The factor $2^{-1/2}$ is included to make comparisons with Brownian motion more convenient and will be made clearer momentarily.

The limiting process globally adopts a parabolic form, but it is locally Brownian—see Figure 1.1. The term “locally Brownian” may be interpreted in several ways, with a progression to stronger forms of interpretation, reflecting recent progress in understanding this limiting scaled profile. “Locally Brownian” could mean that, for any given $x \in \mathbb{R}$, the distributional process limit of $y \mapsto \varepsilon^{-1/2} (\mathcal{L}(x + \varepsilon y) - \mathcal{L}(x))$ as $\varepsilon \searrow 0$ is a standard Brownian motion (where it is the presence of the factor $2^{-1/2}$ in the definition of \mathcal{L} that permits the diffusion rate to equal one, as \mathcal{A} itself is locally