Astérisque
327, 2009, p. 55–68

FEYNMAN INTEGRALS AS HIDA DISTRIBUTIONS: THE CASE OF NON-PERTURBATIVE POTENTIALS

by

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Dedicated to Jean-Michel Bismut as a small token of appreciation

Abstract. — In this note the concepts of path integrals as generalized expectations of White Noise distributions is presented. Combining White Noise techniques with a generalized time-dependent Doss' formula Feynman integrands are constructed as Hida distributions beyond perturbation theory.

***Résumé* (Les intégrales de chemins comme distributions de Hida: le cas de potentiel non-perturbatif)**

Dans cette note, on introduit les intégrales de chemins comme étant des espérances de bruits blancs généralisés. On combine les techniques de bruits blancs avec une généralisation de la méthode de Doss pour construire les « intégrales » de Feynman comme distributions de Hida, au-delà de la théorie perturbative.

1. Introduction

Feynman "integrals", such as

$$J = \int d^{\infty}x \exp\left(i\int_0^t \left(T(\dot{x}(s)) - V\left(x(s)\right)\right) ds\right) f(x(\cdot))$$

are commonplace in physics and meaningless mathematically as they stand. Within white noise analysis [**1, 2, 9, 10, 12, 14, 15, 16, 17**] the concept of integral has a natural extension in the dual pairing of generalized and test functions and allows for the construction of generalized functions (the "Feynman integrands") for various classes of interaction potentials V, see e.g. [**5, 6, 7, 10, 11, 13, 17**], all of them by perturbative methods. This work extends this framework to the case where these fail, using complex scaling as in [**4**], see also [**3**].

In Section 2 we characterize Hida distributions. In Section 3 the U-functional is constructed, see Theorem 3.3. We prove in Section 4 that we obtain a solution of the

2010 Mathematics Subject Classification. — 60H40, 81S40.
Key words and phrases. — Feynman path integrals, white noise analysis.

Schroedinger equation, see Theorem 4.4. The strategy for a general construction of the Feynman integrand is provided in Section 5. Examples are given in Section 6.

2. White Noise Analysis

The white noise measure μ on Schwartz distribution space arises from the characteristic function

$$C(f) := \exp\Big(-\tfrac{1}{2}\|f\|_2^2\Big), \quad f \in S(\mathbb{R}),$$

via Minlos' theorem, see e.g. [**1, 9, 10**]:

$$C(f) = \int_{S'} \exp\big(i\langle \omega, f\rangle\big)\, d\mu(\omega).$$

Here $\langle\cdot,\cdot\rangle$ denotes the dual pairing of $S'(\mathbb{R})$ and $S(\mathbb{R})$. We define the space

$$\big(L^2\big) := L^2(S'(\mathbb{R}), \mathcal{B}, \mu).$$

In the sense of an L^2-limit to indicator functions $\mathbf{1}_{[0,t)}, t > 0$, a version of Wiener's Brownian motion is given by:

$$B(t,\omega) := \langle \omega, \mathbf{1}_{[0,t)}\rangle = \int_0^t \omega(s)\, ds, \quad t > 0.$$

One then constructs a Gel'fand triple:

$$(S) \subset L^2(\mu) \subset (S)'$$

of Hida test functions and distributions, see e.g. [**10**]. We introduce the T-transform of $\Phi \in (S)'$ by

$$(T\Phi)(g) := \big\langle\!\big\langle \Phi, \exp\big(i\langle\cdot, g\rangle\big)\big\rangle\!\big\rangle, \quad g \in S(\mathbb{R}),$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denotes the bilinear dual pairing between $(S)'$ and (S). Expectation extends to Hida distributions Φ by

$$E_\mu(\Phi) := \langle\!\langle \Phi, 1\rangle\!\rangle.$$

Definition 2.1. — *A function $F : S(\mathbb{R}) \to \mathbb{C}$ is called U-functional if*

(i): *F is "ray-analytic": for all $g, h \in S(\mathbb{R})$ the mapping*

$$\mathbb{R} \ni y \mapsto F(g + yh) \in \mathbb{C}$$

has an analytic continuation to $\mathbb{C}$ as an entire function.

(ii): *F is uniformly bounded of order 2, i.e., there exist some constants $0 < K, D < \infty$ and a continuous norm $\|\cdot\|$ on $S(\mathbb{R})$ such that for all $w \in \mathbb{C}$, $g \in S(\mathbb{R})$*

$$|F(wg)| \leq K \exp(D|w|^2\|g\|^2).$$

Theorem 2.2. — *The following statements are equivalent:*

(i): *$F : S(\mathbb{R}) \to \mathbb{C}$ is a U-functional.*

(ii): *F is the T-transform of a unique Hida distribution $\Phi \in (S)'$.*

For the proof and more see e.g. [**10**].

3. Hida distributions as candidates for Feynman Integrands

In this section we construct Hida distributions as candidates for the Feynman integrands. First we list which properties potentials must fulfill.

Assumption 3.1. — *For $\mathcal{O} \subset \mathbb{R}$ open, where $\mathbb{R} \setminus \mathcal{O}$ is a set of Lebesgue measure zero, we define the set $\mathcal{D} \subset \mathbb{C}$ by*

$$\mathcal{D} := \left\{ x + \sqrt{i} y \;\middle|\; x \in \mathcal{O} \text{ and } y \in \mathbb{R} \right\},$$

and consider analytic functions $V_0 : \mathcal{D} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$. Let $0 \leq t \leq T < \infty$. We require that there exists an $0 < \varepsilon < 1$ and a function $I : \mathcal{D} \to \mathbb{R}$ such that its restriction to $\mathcal{O}$ is measurable and locally bounded and

(3.1)

$$E\left[\left| \exp\left(-i \int_0^t V_0\left(z + \sqrt{i} B_s\right) ds \right) f\left(z + \sqrt{i} B_t\right) \right| \exp\left(\frac{\varepsilon \|B\|_{\sup,T}^2}{2} \right) \right] \leq I(z), \quad z \in \mathcal{D},$$

uniformly in $0 \leq t \leq T$. Here E denotes the expectation w.r.t. a Brownian motion B starting at 0. $\| \cdot \|_{\sup,T}$ denotes the supremum norm over $[0, T]$.

We shall consider time-dependent potentials of the form

$$\begin{aligned} V_{\dot{g}} : [0,T] \times \mathcal{D} &\to \mathbb{C} \\ (t, z) &\mapsto V_0(z) + \dot{g}(t) z \end{aligned} \tag{3.2}$$

for $g \in S(\mathbb{R})$.

Remark 3.2. — *One can show that (3.1) implies that*

$$E\left[\exp\left(-i \int_0^{t-t_0} V_{\dot{g}}\left(t - s, z + \sqrt{i} B_s\right) ds \right) f\left(z + \sqrt{i} B_{t-t_0}\right) \right],$$

is well-defined for all $g \in S(\mathbb{R})$, $0 \leq t_0 \leq t \leq T$ and $z \in \mathcal{D}$.

Theorem 3.3. — *Let $0 < T < \infty$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be Borel measurable, bounded with compact support. Moreover we assume that V_0 and f fulfill Assumption 3.1. Then for all $0 \leq t_0 \leq t \leq T$, the mapping*

$$\begin{aligned} &F_{\varphi,t,f,t_0} : S(\mathbb{R}) \to \mathbb{C} \\ &g \mapsto \exp\left(-\frac{1}{2} \int_{[t_0,t]^c} g^2(s)\, ds \right) \int_{\mathbb{R}} \exp(-ig(t_0)x) \varphi(x) \Big(G(g,t,t_0) \exp(ig(t)\cdot) f \Big)(x)\, dx \end{aligned} \tag{3.3}$$

is a U-functional where for $x \in \mathcal{O}$

$$(3.4)\quad \Big(G(g,t,t_0)\exp(ig(t)\cdot)f\Big)(x) := E\Bigg[\exp\Bigg(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, x+\sqrt{i}B_s\Big)\,ds\Bigg) \times \exp\Big(ig(t)\Big(x+\sqrt{i}B_{t-t_0}\Big)\Big) f\Big(x+\sqrt{i}B_{t-t_0}\Big)\Bigg].$$

Proof. — F_{φ,t,f,t_0} is well-defined: (3.4) is finite because of (3.1), and the integral in (3.3) exists since φ is bounded with compact support.

To show that F_{φ,t,f,t_0} is a U-functional we must verify two properties, see Definition 2.1.

First F_{φ,t,f,t_0} must have a "ray-analytic" continuation to $\mathbb{C}$ as an entire function. I.e., for all $g, h \in S(\mathbb{R})$ the mapping

$$\mathbb{R} \ni y \mapsto F_{\varphi,t,f,t_0}(g+yh) \in \mathbb{C}$$

has an entire extension to $\mathbb{C}$.

We note first that this is true for the expression

$$(3.5)\quad u(y) := \exp\Bigg(-i\int_0^{t-t_0} V_{\dot g + y\dot h}\Big(t-s, x+\sqrt{i}B_s\Big)\,ds\Bigg) \times \exp\Big(i\,(g+yh)\,(t)\Big(x+\sqrt{i}B_{t-t_0}\Big)\Big) f\Big(x+\sqrt{i}B_{t-t_0}\Big)$$

inside the expectation in (3.4). Hence the integral of u over any closed curve in $\mathbb{C}$ is zero. By Lebesgue dominated convergence the expectation $E[u(w)]$ is continuous in w. With Fubini

$$\oint E\left[u\left(w\right)\right] dw = E\left[\oint u\left(w\right) dw\right] = 0,$$

for all closed paths, hence by Morera $E(u(w))$ is entire. This extends to (3.3) since φ is bounded with compact support. Thus

$$\mathbb{C} \ni w \mapsto F_{\varphi,t,f,t_0}(g+wh) \in \mathbb{C}$$

is entire for all $0 \leq t_0 \leq t \leq T$ and all $g, h \in S(\mathbb{R})$.

Verification is straightforward that F_{φ,t,f,t_0} is of 2nd order exponential growth, F_{φ,t,f,t_0} is a U-functional. □

One can show the same result by choosing the delta distribution $\delta_x,\ x \in \Theta$, instead of a test function φ:

Corollary 3.4. — *Let V_0 and f fulfill Assumption 3.1 and let $x \in \Theta$. Then for all $0 \leq t_0 \leq t \leq T$ the mapping*

$$F_{\delta_x,t,f,t_0} : S(\mathbb{R}) \to \mathbb{C}$$

$$g \mapsto \exp\Bigg(-\frac{1}{2}\int_{[t_0,t]^c} g^2(s)\,ds\Bigg)\exp(-ig(t_0)x)\Big(G(g,t,t_0)\exp(ig(t)\cdot)\Big)f(x)$$

is a U-functional, where $\Big(G(g,t,t_0)\exp(ig(t)\cdot)\Big)f(x)$ *is defined as in Theorem 3.3.*

4. Solution to time-dependent Schrödinger equation

Assumption 4.1. — *Let* $V_0 : \mathcal{D} \to \mathbb{C}$ *and* $f : \mathbb{C} \to \mathbb{C}$ *such that Assumption 3.1 is fulfilled and* $V_{\dot g}$, $g \in S(\mathbb{R})$, *as in (3.2).*

(i): *For all* $u, v, r, l \in [0, T]$ *and all* $z \in \mathcal{D}$ *we require that*

$$E^1\Bigg[\Bigg|\exp\bigg(-i\int_0^u V_{\dot g}\Big(v-s, z+\sqrt{i}B^1_s\Big)\,ds\bigg)$$

$$\times E^2\Bigg[\exp\bigg(-i\int_0^r V_{\dot g}\Big(l-s, z+\sqrt{i}B^1_u+\sqrt{i}B^2_s\Big)\,ds\bigg) f\Big(z+\sqrt{i}B^1_u+\sqrt{i}B^2_r\Big)\Bigg]\Bigg|\Bigg] < \infty. \tag{4.1}$$

(ii): *For all* $z \in \mathcal{D}$, $0 \le t_0 \le t \le T$ *and some* $0 < \varepsilon \le T$ *the functions*

$$\omega \mapsto \sup_{0\le h\le\varepsilon}\Bigg|\bigg(V_{\dot g}\Big(t, z+\sqrt{i}B_h(\omega)\Big)+\int_0^h \frac{\partial}{\partial t}V_{\dot g}\Big(t+h-s, z+\sqrt{i}B_s(\omega)\Big)\,ds\bigg)$$

$$\times\exp\bigg(-i\int_0^h V_{\dot g}\Big(t+h-s, z+\sqrt{i}B_s(\omega)\Big)\,ds\bigg) f\Big(z+\sqrt{i}B_h(\omega)\Big)\Bigg| \tag{4.2}$$

and

$$\omega \mapsto \sup_{h\in[0,T]}\Bigg|\Delta E^2\Bigg[\exp\bigg(-i\int_0^{t-t_0} V_{\dot g}\Big(t-s, z+\sqrt{i}B^1_h(\omega)+\sqrt{i}B^2_s\Big)\,ds\bigg)$$

$$\times f\Big(z+\sqrt{i}B^1_h(\omega)+\sqrt{i}B^2_{t-t_0}\Big)\Bigg]\Bigg| \tag{4.3}$$

are integrable.

Here B^1 *and* B^2 *are Brownian motions starting at 0 with corresponding expectations* E^1 *and* E^2, *respectively. Moreover* Δ *denotes* $\frac{\partial^2}{\partial z^2}$ *and* $\frac{\partial}{\partial t}$ *the derivative w.r.t. the first variable.*

We define $H(\mathcal{D})$ to be the set of holomorphic functions from $\mathcal{D}$ to $\mathbb{C}$. As pointed out by H. Doss, see [**4**], under specified assumptions (similar to Assumption 3.1 and Assumption 4.1 (ii)) there is a solution $\psi : [0,T]\times\mathcal{D} \to \mathbb{C}$ to the time-independent Schrödinger equation, i.e., for all $t \in [0,T]$ and $x \in \mathcal{O}$

$$\begin{cases} i\frac{\partial}{\partial t}\psi(t,x) = -\frac{1}{2}\Delta\psi(t,x)+V_0(x)\psi(t,x) \\ \psi(0,x) = f(x), \end{cases}$$

which is given by

$$\psi(t,x) = E\Bigg[\exp\bigg(-i\int_0^t V_0\Big(x+\sqrt{i}B_s\Big)ds\bigg) f\Big(x+\sqrt{i}B_t\Big)\Bigg].$$