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SMOOTH DENSITY OF CANONICAL STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS

by

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Dedicated to Professor J.-M. Bismut for his sixtieth birthday

Abstract. — We consider jump diffusion process ξ_t on $\mathbf{R}^d$ determined by a canonical SDE:

$$d\xi_t = \sum_{i=1}^m V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt,$$

where $Z_t = (Z_t^1, ..., Z_t^m)$ is an m-dimensional Lévy process and $V_0, ..., V_m$ are smooth vector fields. We prove that the law of the solution ξ_t has a C^∞-density under the following two conditions. (1) The Lévy process Z_t is nondegenerate. (2) $\{V_0, V_1, ..., V_m\}$ can be degenerate but satisfies a uniform Hörmander condition (H). For the proof we make use of the Malliavin calculus on the Wiener-Poisson space studied by Ishikawa-Kunita.

Résumé **(Densité lisse pour les solutions d'équations différentielles stochastiques avec sauts)**

Nous considérons un processus de diffusion à sauts ξ_t dans $\mathbf{R}^d$ déterminé par une EDS canonique:

$$d\xi_t = \sum_{i=1}^m V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt,$$

où $Z_t = (Z_t^1, ..., Z_t^m)$ est un processus de Lévy m-dimensionnel et $V_0, ..., V_m$ sont des champs de vecteurs. Nous montrons que la loi de ξ_t a une densité C^∞ si les conditions suivantes sont satisfaites. (1) Le processus de Lévy Z_t est non dégénéré. (2) La distribution $\{V_0, V_1, ..., V_m\}$ peut être dégénérée mais elle satisfait à une condition de Hörmander uniforme (H). Pour la démonstration, nous utilisons le calcul de Malliavin sur l'espace de Wiener-Poisson étudié par Ishikawa-Kunita.

1. Introduction and main results

Let $V_0, V_1, \cdots V_m$ be smooth vector fields on $\mathbf{R}^d$ whose derivatives (including higher orders) are all bounded. Let $Z_t = (Z_t^1, ..., Z_t^m), t \geq 0$ be an m-dimensional nondegenerate Lévy process. In this paper, we consider a jump diffusion determined by a

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canonical SDE based on $\{V_0, V_1, \cdots, V_m\}$ and Z_t;

$$d\xi_t = \sum_{i=1}^{m} V_i(\xi_t) \diamond dZ_t^i + V_0(\xi_t)dt. \tag{1.1}$$

Canonical SDE's are studied in mathematical finance. Let Z_t be a one dimensional Lévy process. We consider a one dimensional linear canonical SDE.

$$dS_t = S_t \diamond dZ_t.$$

The solution starting from S_0 at time 0 is unique and it is written as $S_t := S_0 \exp Z_t$ (See Section 2). It is called a *geometric Lévy process*. The solution S_t describes the movement of a stock. If Z_t is a Lévy process with finite Lévy measure (a compound Poisson process), the process S_t is the *Merton model* or the *Kou model*, according as the normalized Lévy measure is a Gaussian distribution or a double exponential distribution, respectively. See [**16**],[**8**]. The precise definition of the canonical SDE will be given at Section 2.

The main purpose of this paper is to show the existence of the smooth density for the law of the random variable ξ_t that is a solution of equation (1.1). For this purpose we need to assume suitable nondegenerate conditions both for the Lévy process Z_t and the family of vector fields $\{V_0, ..., V_m\}$.

We first consider the Lévy process. The Lévy process Z_t is represented for arbitrary $\delta > 0$, by

$$Z_t = \sigma W_t + \int_0^t \int_{0<|z|\le\delta} z\tilde{N}(drdz) + \int_0^t \int_{|z|>\delta} zN(drdz) + b_\delta t,$$

where σ is an $m \times m$-matrix, W_t is an m-dimensional standard Brownian motion. $N(dtdz)$ is a Poisson random measure which is independent of W_t with intensity $\hat{N}(dtdz) = dt\nu(dz)$, ν being the Lévy measure. Further, $\tilde{N}(dtdz) = N(dtdz) - \hat{N}(dtdz)$ and $b_\delta = (b_\delta^1, ..., b_\delta^m)$ is a drift vector. Set $A = (a_{ij}) = \sigma\sigma^T$. It is a covariance of the Gaussian part σW_1 (Lévy-Itô decomposition). Throughout this paper, we assume that the Lévy measure ν has finite moments of any order. Set $v(\rho) := \int_{|z|<\rho} |z|^2 \nu(dz)$. If there exists $\alpha \in (0, 2)$ such that

$$\liminf_{\rho \to 0} \frac{v(\rho)}{\rho^\alpha} > 0,$$

then the Lévy measure is said to satisfy an *order condition*. Note that the Lévy measure ν satisfying an order condition is an infinite measure: Indeed, we have $\nu(\{z; 0 < |z| < \delta\}) = \infty$ for any $\delta > 0$. In case of one dimensional Lévy process, the above order condition is known as a sufficient condition for the existence of the smooth density of the law of the Lévy process (Orey's theorem. See Sato [**20**], Proposition 28.3). Then the law of the geometric Lévy process S_t has a smooth density if the order condition is satisfied.

Now we set $b_{ij}(\rho) = \int_{|z|\le\rho} z^i z^j \nu(dz)/v(\rho)$ and $B(\rho) = (b_{ij}(\rho))$. The *infinitesimal covariance* B is a symmetric and nonnegative definite matrix, which coincides with the greatest lower bound of the matrix $B(\rho)$ as $\rho \to 0$. If the Lévy measure satisfies an order condition and the matrix $A + B$ is nondegenerate (invertible), then we say that the Lévy process is *nondegenerate.* In this paper, *we assume that the Lévy process Z_t is nondegenerate.*

We will next consider nondegenete properties for the family of vector fields $\{V_0, ..., V_m\}$. In Ishikawa-Kunita **[6]**, we studied the case where the family of vector fields $\{V_1, ..., V_m\}$ is *uniformly nondegenerate*, i.e., there exists a positive constant C such that the inequality

$$\sum_{i=1}^{m} |l^T V_i(x)|^2 \ge C|l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^d$$

holds valid, where l^T is the transpose of l and $l^T V(x)$ denotes the inner product of two vectors l and $V(x)$. We showed the existence of the smooth density of its law by applying Malliavin calculus on the Wiener-Poisson space.

In this paper we want to relax the above uniformly nondegenerate condition. Let $V_0, ..., V_m$ be C^∞-vector fields such that their derivatives (including higher orders) are all bounded. Then Lie brackets $[V_{i_1}[\cdots[V_{i_{n-1}}, X_{i_n}]\cdots]$, $i_1, ..., i_n \in \{0, 1, ..., m\}$ are bounded vector fields. We introduce families of vector fields. Let $\Sigma_0 = \{V_1, ..., V_m\}$ be a linear space of vector fields spanned by $V_1, ..., V_m$. Given $\delta > 0$, we set

$$\hat{V}_0^\delta = V_0 + \sum_{i=1}^{m} b_\delta^i V_i.$$

Set $\Sigma_0^\delta = \Sigma_0$ and define for $k = 1, 2, ...$

$$\Sigma_k^\delta = \left\{ [\hat{V}_0^\delta, V] + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}[V_i, [V_j, V]],\ [V_i, V], i = 1, ..., m, V \in \Sigma_{k-1}^\delta \right\}.$$

Theorem 1.1. — *Assume that for the family of vector fields $\{V_0, ..., V_m\}$ there exist a positive integer N_0 and a positive number δ_0 such that for any $0 < \delta < \delta_0$ the inequality*

$$\sum_{k=0}^{N_0} \sum_{V \in \Sigma_k^\delta} |l^T V(x)|^2 \ge C(\delta)|l|^2, \quad \forall x \in \mathbf{R}^d, \quad \forall l \in \mathbf{R}^d \tag{1.2}$$

holds valid, where $C(\delta)$ are positive numbers satisfying

$$\liminf_{\delta \to 0} C(\delta)/v(\delta)^2 = \infty.$$

Then for any initial random variable ξ_0 and $0 < T_0 < \infty$, the law of the solution ξ_{T_0} of the canonical SDE (1.1) has a C^∞-density.

The condition required for vector fields in the above theorem is complicated, since δ's are involved. We can replace it by a simpler one if we restrict the Lévy process Z_t to a simpler one, namely if we assume

$$b_0 = \lim_{\delta \to 0} b_\delta \quad \text{exists and is finite.} \tag{1.3}$$

The existence of b_0 is equivalent to that of $\lim_{\delta\to 0} \int_{\delta<|z|\le 1} z\nu(dz)$. In this case, it holds $b_0 = b_1 - \lim_{\delta\to 0}\int_{\delta<|z|\le 1} z\nu(dz)$. In particular, if the integral $\int_{0<|z|\le 1}|z|\nu(dz)$ is finite, b_0 exists and is finite. Hence for any stable process whose exponent is less than 1, b_0 exists. Further, if the Lévy measure ν is symmetric, b_0 exists and is equal to b_1 even if $\int_{0<|z|\le 1}|z|\nu(dz)$ is infinite. Hence for any symmetric stable process, b_0 exists and is equal to b_1.

Now, assume (1.3) and let $\delta \to 0$ in the Lévy-Itô decomposition of Z_t. Then we obtain

$$Z_t = \sigma W_t + \int_0^t \int_{|z|>0} zN(drdz) + b_0 t.$$

Hence b_0 can be regarded as the drift vector of the Lévy process Z_t. We define a new drift vector field $\hat{V}_0$ by

$$\hat{V}_0 = V_0 + \sum_{i=1}^m b_0^i V_i,$$

and introduce families of vector fields by $\Sigma_0 = \{V_1, ..., V_m\}$ and for $k = 1, ...$

$$\Sigma_k = \Big\{ [\hat{V}_0, V] + \frac{1}{2}\sum_{i,j=1}^m a_{ij}[V_i,[V_j,V]],\ [V_i,V], i = 1,...,m, V\in\Sigma_{k-1}\Big\}.$$

Theorem 1.2. — *Assume (1.3) for the Lévy process Z_t. Assume further that the family of vector fields $\{\hat{V}_0, V_1, ..., V_m\}$ satisfy the uniform Hörmander condition (H), i.e., there exists a positive integer N_0 and a positive constant C such that*

$$\sum_{k=0}^{N_0}\sum_{V\in\Sigma_k} |l^T V(x)|^2 \ge C|l|^2, \quad \forall x\in\mathbf{R}^d, \quad \forall l \in \mathbf{R}^d \tag{1.4}$$

holds valid. Then for any initial random variable ξ_0 and $0 < T_0 < \infty$, the law of the solution ξ_{T_0} of the canonical SDE (1.1) has a C^∞-density.

Observe that Theorem 1.2 indicates that both the canonical SDE with jumps and Stratonovich SDE (diffusion) have the common local criterion (Hörmander' condition) for the existence of the smooth density of their laws. This is partly because that we restrict our attention to small jumps of the SDE, ignoring the effect of big jumps. Loosely speaking, under an order condition, the solution of equation (1.1) could behave like a diffusion if sizes of jumps are small.

Perhaps, Bismut **[2]** is the first work toward the smooth density of the law of the solution of SDE with jumps, where he developed the Malliavin calculus for jump processes. After this fundamental work, the similar problem has been discussed in some different contexts by Léandre **[13]**,**[14]**,**[15]**, Bichteler-Gravreau-Jacod **[1]**, Komatsu-Takeuchi **[7]** and others. A common feature in the above works might be that they assumed for the Lévy measure ν the existence of a smooth density and an asymptotic of the density as $z \to 0$. Furthermore, a formula of integration by parts holds valid in these cases, which are shown through Girsanov's theorem for jump diffusion.

In our discussion any Lévy measure (singular or not) is allowed, as far as it satisfies an order condition. Then no formula of integration by parts is known. We take another approach to the Malliavin calculus, developed in Ishikawa-Kunita **[6]**. It will be presented in the next section.

2. Malliavin calculus for canonical SDE

Let $Z_t, t \geq 0$ be an m-dimensional Lévy process admitting the Lévy-Itô decomposition and let ξ_0 be an $\mathbf{R}^d$-valued random variable independent of Z_t. By the solution of equation (1.1) starting from ξ_0 at time 0, we mean a cadlag $\mathbf{R}^d$-valued semimartingale $\{\xi_t; t \geq 0\}$ adapted to $\mathscr{F}_t = \sigma(\xi_0, Z_r; r \leq t)$ satisfying

$$
\begin{aligned}
\xi_t &= \xi_0 + \sum_{i=1}^m \int_0^t V_i(\xi_r) \diamond dZ_r^i + \int_0^t V_0(\xi_r) dr \\
&= \xi_0 + \sum_{i,k=1}^m \int_0^t V_i(\xi_r)\sigma_{ik} \circ dW_r^k + \int_0^t \hat{V}_0^\delta(\xi_r) dr. \\
&\quad + \int_0^t \int_{|z|<\delta} \{\phi_1^z(\xi_{r-}) - \xi_{r-}\} \tilde{N}(drdz) \\
&\quad + \int_0^t \int_{|z|\geq\delta} \{\phi_1^z(\xi_{r-}) - \xi_{r-}\} N(drdz) \\
&\quad + \int_0^t \int_{|z|<\delta} \{\phi_1^z(\xi_r) - \xi_r - \sum_{i=1}^m z^i V_i(\xi_r)\} \hat{N}(drdz).
\end{aligned} \tag{2.1}
$$

Here " $\circ$ " denotes the Stratonovitch integral. Using Itô integral, it holds

$$
\begin{aligned}
&\sum_{k=1}^m \int_0^t V_i(\xi_r)\sigma_{ik} \circ dW_r^k \\
&= \sum_{k=1}^m \int_0^t V_i(\xi_{r-})\sigma_{ik} dW_r^k + \frac{1}{2}\sum_{j=1}^m a_{ij} \int_0^t \Big(\sum_{l=1}^d \frac{\partial V_i}{\partial x^l} V_j^l\Big)(\xi_{r-}) dr.
\end{aligned}
$$

Further, for $z = (z^1, ..., z^m) \in \mathbf{R}^m$ $\phi_s^z, s \in \mathbf{R}$ is the one parameter group of diffeomorphisms generated by the vector field $\sum_{i=1}^m z^i V_i$, i.e., $\phi_s^z = \exp s(\sum_i z^i V_i)$.