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(1009) *Regularity of optimal transport maps*

Alessio FIGALLI

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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REGULARITY OF OPTIMAL TRANSPORT MAPS [after Ma–Trudinger–Wang and Loeper]

by **Alessio FIGALLI**

INTRODUCTION

In the field of optimal transportation, one important issue is the regularity of the optimal transport map. There are several motivations for the investigation of the smoothness of the optimal map:

- It is a typical PDE/analysis question.
- It is a step towards a qualitative understanding of the optimal transport map.
- If it is a general phenomenon, then non-smooth situations may be treated by regularization, instead of working directly on non-smooth objects.

In the special case "cost = squared distance" on $\mathbb{R}^n$, the problem was solved by Caffarelli [**4, 5, 6, 7**], who proved the smoothness of the map under suitable assumptions on the regularity of the densities and on the geometry of their support. However, a major open problem in the theory was the question of regularity for more general cost functions, or for the case "cost = squared distance" on a Riemannian manifold. A breakthrough in this problem has been achieved by Ma, Trudinger and Wang [**27**] and Loeper [**24**], who found a necessary and sufficient condition on the cost function in order to ensure regularity. This condition, now called MTW condition, involves a combination of derivatives of the cost, up to the fourth order. In the special case "cost = squared distance" on a Riemannian manifold, the MTW condition corresponds to the non-negativity of a new curvature tensor on the manifold (the so-called MTW tensor), which implies strong geometric consequences on the geometry of the manifold and on the structure of its cut-locus.

1. THE OPTIMAL TRANSPORTATION PROBLEM

The Monge transportation problem is more than 200 years old [**29**], and it has generated a huge amount of work in the last years.

Originally Monge wanted to move, in the Euclidean space $\mathbb{R}^3$, a rubble (*déblais*) to build up a mound or fortification (*remblais*) minimizing the cost. To explain this in a simple case, suppose that the rubble consists of masses, say $m_1, \dots, m_n$, at locations $\{x_1, \dots x_n\}$, and one is interested in moving them into another set of positions $\{y_1, \dots, y_n\}$ by minimizing the weighted travelled distance. Then, one tries to minimize

$$\sum_{i=1}^{n} m_i |x_i - T(x_i)|,$$

over all bijections $T : \{x_1, \dots x_n\} \to \{y_1, \dots, y_n\}$.

Nowadays, influenced by physics and geometry, one would be more interested in minimizing the energy cost rather than the distance. Therefore, one wants to minimize

$$\sum_{i=1}^{n} m_i |x_i - T(x_i)|^2.$$

Of course, it is desirable to generalize this problem to continuous, rather than just discrete, distributions of matter. Hence, the optimal transport problem is now formulated in the following general form: given two probability measures μ and ν, defined on the measurable spaces X and Y, find a measurable map $T : X \to Y$ with $T_\sharp\mu = \nu$, i.e.

$$\nu(A) = \mu\left(T^{-1}(A)\right) \qquad \forall\, A \subset Y \text{ measurable},$$

in such a way that T minimizes the transportation cost. This means

$$\int_X c(x, T(x))\, d\mu(x) = \min_{S_\# \mu = \nu} \left\{ \int_X c(x, S(x))\, d\mu(x) \right\},$$

where $c : X \times Y \to \mathbb{R}$ is some given cost function, and the minimum is taken over all measurable maps $S : X \to Y$ such that $S_\#\mu = \nu$. When the transport condition $T_\#\mu = \nu$ is satisfied, we say that T is a *transport map*, and if T also minimizes the cost we call it an *optimal transport map*.

Even in Euclidean spaces, with the cost c equal to the Euclidean distance or its square, the problem of the existence of an optimal transport map is far from being trivial. Moreover, it is easy to build examples where the Monge problem is ill-posed simply because there is no transport map: this happens for instance when μ is a Dirac mass while ν is not. This means that one needs some restrictions on the measures μ and ν.

We further remark that, if $\mu(dx) = f(x)dx$ and $\nu(dy) = g(y)dy$, the condition $T_\#\mu = \nu$ formally gives the Jacobian equation $\left|\det(\nabla T)\right| = f/(g \circ T)$.

1.1. Existence and uniqueness of optimal maps on Riemannian manifolds

In [**1, 2**], Brenier considered the case $X = Y = \mathbb{R}^n$, $c(x,y) = |x-y|^2/2$, and he proved the following theorem (the same result was also proven independently by Cuesta-Albertos and Matrán [**10**] and by Rachev and Rüschendorf [**30**]):

THEOREM 1.1 ([**1, 2**]). — *Let μ and ν be two compactly supported probability measures on $\mathbb{R}^n$. If μ is absolutely continuous with respect to the Lebesgue measure, then:*

(i) *There exists a unique solution T to the Monge problem.*
(ii) *The optimal map T is characterized by the structure $T(x) = \nabla\phi(x)$, for some convex function $\phi : \mathbb{R}^n \to \mathbb{R}$.*

Furthermore, if $\mu(dx) = f(x)dx$ and $\nu(dy) = g(y)dy$,

$$\big|\det(\nabla T(x))\big| = \frac{f(x)}{g(T(x))} \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

After this result, many researchers started to work on the problem, showing existence of optimal maps with more general costs, both in an Euclidean setting, in the case of compact (Riemannian and sub-Riemannian) manifolds, and in some particular classes on non-compact manifolds. In particular, exploiting some ideas introduced by Cabré in [**3**] for studying elliptic equations on manifolds, McCann was able to generalize Brenier's theorem to (compact) Riemannian manifolds [**28**].

REMARK. — *From now on, we will always implicitly assume that all manifolds have no boundary.*

To explain McCann's result, let us first introduce a few definitions.

We recall that a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous if and only if

$$\varphi(x) = \sup_{y\in\mathbb{R}^n} \big[x\cdot y - \varphi^*(y)\big],$$

where

$$\varphi^*(x) := \sup_{x\in\mathbb{R}^n} \big[x\cdot y - \varphi(x)\big].$$

This fact is the basis for the notion of *c-convexity*, where $c : X \times Y \to \mathbb{R}$ is an arbitrary function:

DEFINITION 1.2. — *A function $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is* c-convex *if*

$$\psi(x) = \sup_{y\in Y}\big[\psi^c(y) - c(x,y)\big] \qquad \forall\, x \in X,$$

where

$$\psi^c(y) := \inf_{x\in X}\big[\psi(x) + c(x,y)\big] \qquad \forall\, y \in Y.$$

Moreover, for a c-convex function ψ, we define its c-subdifferential *at x as*

$$\partial^c\psi(x) := \big\{y \in Y \mid \psi(x) = \psi^c(y) - c(x,y)\big\}.$$

With this general definition, when $X = Y = \mathbb{R}^n$ and $c(x,y) = -x\cdot y$, the usual convexity coincides with the c-convexity, and the usual subdifferential coincides with the c-subdifferential.

In particular, in the case $X = Y = \mathbb{R}^n$ and $c(x,y) = |x-y|^2/2$, a function ψ is c-convex if and only if $\psi(x) + \frac{|x|^2}{2}$ is convex. The following result is the generalization of Brenier's Theorem to Riemannian manifolds:

THEOREM 1.3 ([**28**]). — *Let (M,g) be a Riemannian manifold, take μ and ν two compactly supported probability measures on M, and consider the optimal transport problem from μ to ν with cost $c(x,y) = d(x,y)^2/2$, where $d(x,y)$ denotes the Riemannian distance on M. If μ is absolutely continuous with respect to the volume measure, then:*

- (i) *There exists a unique solution T to the Monge problem.*
- (ii) *T is characterized by the structure $T(x) = \exp_x\big(\nabla\psi(x)\big) \in \partial^c\psi(x)$ for some c-convex function $\psi : M \to \mathbb{R}$.*
- (iii) *For μ_0-a.e. $x \in M$, there exists a unique minimizing geodesic from x to $T(x)$, which is given by $[0,1] \ni t \mapsto \exp_x\big(t\nabla\psi(x)\big)$.*

Furthermore, if $\mu(dx) = f(x)\mathrm{vol}(dx)$ and $\nu(dy) = g(y)\mathrm{vol}(dy)$,

$$\big|\det(\nabla T(x))\big| = \frac{f(x)}{g(T(x))} \qquad \text{for } \mu\text{-a.e. } x \in M.$$

The last formula in the above theorem needs a comment: given a function $T : M \to M$, the determinant of its Jacobian is not intrinsically defined. Indeed, in order to compute the determinant of $\nabla T(x) : T_xM \to T_{T(x)}M$, one needs to identify the tangent spaces. On the other hand, $\big|\det(\nabla T(x))\big|$ is intrinsically defined as

$$\big|\det(\nabla T(x))\big| = \lim_{r\to 0} \frac{\mathrm{vol}(T(B_r(x)))}{\mathrm{vol}(B_r(x))},$$

whenever the above limit exists.

2. THE REGULARITY ISSUE: THE EUCLIDEAN CASE

Let Ω and Ω' be two bounded smooth open sets in $\mathbb{R}^n$, and let $\mu(dx) = f(x)dx$, $\nu(y) = g(y)dy$ be two probability measures, with f and g such that $f = 0$ in $\mathbb{R}^2 \setminus \Omega$, $g = 0$ in $\mathbb{R}^2 \setminus \Omega'$. We assume that f and g are C^∞ and bounded away from zero and infinity on Ω and Ω', respectively. By Brenier's Theorem, the optimal transport