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# THE JACOBIAN MAP, THE JACOBIAN GROUP AND THE GROUP OF AUTOMORPHISMS OF THE GRASSMANN ALGEBRA

BY VLADIMIR V. BAVULA

ABSTRACT. — There are nontrivial dualities and parallels between polynomial algebras and the Grassmann algebras (e.g., the Grassmann algebras are dual of polynomial algebras as quadratic algebras). This paper is an attempt to look at the Grassmann algebras at the angle of the Jacobian conjecture for polynomial algebras (which is the question/conjecture about the *Jacobian set* – the set of all algebra endomorphisms of a polynomial algebra with the Jacobian 1 – the Jacobian conjecture claims that the Jacobian set is a *group*). In this paper, we study in detail the Jacobian set for the Grassmann algebra which turns out to be a *group* – the *Jacobian group* $\Sigma$ – a sophisticated (and large) part of the group of automorphisms of the Grassmann algebra $\Lambda_n$. It is proved that the Jacobian group $\Sigma$ is a rational unipotent algebraic group. A (minimal) set of generators for the algebraic group $\Sigma$, its dimension and coordinates are found explicitly. In particular, for $n \geq 4$,

$$\dim(\Sigma) = \begin{cases} (n-1)2^{n-1} - n^2 + 2 & \text{if } n \text{ is even,} \\ (n-1)2^{n-1} - n^2 + 1 & \text{if } n \text{ is odd.} \end{cases}$$

The same is done for the Jacobian ascents - some natural algebraic overgroups of $\Sigma$. It is proved that the Jacobian map $\sigma \mapsto \det(\frac{\partial \sigma(x_i)}{\partial x_j})$ is surjective for odd $n$, and is *not* for even $n$ though, in this case, the image of the Jacobian map is an algebraic subvariety of codimension 1 given by a single equation.

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Résumé (*L'application de Jacobi, le groupe de Jacobi et le groupe des automorphismes de l'algèbre grassmanienne*)

Il existe des dualités et des parallélismes non-triviaux entre les algèbres polynomiales et les algèbres grassmaniennes (par ex., les algèbres grassmaniennes sont duales des algèbres polynomiales en tant qu'algèbres quadratiques). Cet article est une tentative d'étude des algèbres grassmaniennes du point de vue de la conjecture de Jacobi sur les algèbres polynomiales (qui est la question/conjecture sur l'*ensemble de Jacobi* — l'ensemble de tous les endomorphismes d'algèbre d'une algèbre polynomiale avec jacobien 1 —, la conjecture de Jacobi affirme que l'ensemble de Jacobi est un *groupe*. Dans cet article nous étudions en détail l'ensemble de Jacobi pour l'algèbre grassmanienne qui s'avère être un *groupe* — le *groupe de Jacobi* $\Sigma$ —, une partie grande et sophistiquée du groupe d'automorphismes de l'algèbre grassmanienne $\Lambda_n$. Nous démontrons que le groupe de Jacobi $\Sigma$ est un groupe algébrique rationnel unipotent. Nous calculons explicitement un ensemble (minimal) de générateurs pour le groupe algébrique $\Sigma$, sa dimension et ses coordonnées. En particulier, pour $n \geq 4$,

$$\dim(\Sigma) = \begin{cases} (n-1)2^{n-1} - n^2 + 2 & \text{si } n \text{ est pair,} \\ (n-1)2^{n-1} - n^2 + 1 & \text{si } n \text{ est impair.} \end{cases}$$

Nous faisons de même pour les ascendants jacobiens — certains surgroupes algébriques naturels de $\Sigma$. Nous démontrons que l'application de Jacobi $\sigma \mapsto \det(\frac{\partial \sigma(x_i)}{\partial x_j})$ est surjective pour $n$ impair, et ne l'est *pas* pour $n$ pair, néanmoins, dans ce cas, l'image d'une application de Jacobi est une sous-variété algébrique de codimension 1, donnée par une seule équation.

## 1. Introduction

Throughout, ring means an associative ring with 1. Let $K$ be an arbitrary ring (not necessarily commutative). The *Grassmann algebra* (the *exterior algebra*) $\Lambda_n = \Lambda_n(K) = K\lfloor x_1, \dots, x_n \rfloor$ is generated freely over $K$ by elements $x_1, \dots, x_n$ that satisfy the defining relations:

$$x_1^2 = \cdots = x_n^2 = 0 \text{ and } x_i x_j = -x_j x_i \text{ for all } i \neq j.$$

**What is the paper about? Motivation. —** Briefly, for the Grassmann algebra $\Lambda_n$ over a commutative ring $K$ we study in detail the *Jacobian map*

$$\mathcal{J}(\sigma) := \det\left(\frac{\partial \sigma(x_i)}{\partial x_j}\right)$$

which is a 'straightforward' generalization of the usual Jacobian map $\mathcal{J}(\sigma) := \det(\frac{\partial \sigma(x_i)}{\partial x_j})$ for a polynomial algebra $P_n = K[x_1, \dots, x_n]$, $\sigma \in \mathrm{End}_{K-alg}(P_n)$. The polynomial Jacobian map is not yet a well-understood map, one of the open questions about this map is the Jacobian conjecture (JC) which claims *that* $\mathcal{J}(\sigma) = 1$ *implies* $\sigma \in \mathrm{Aut}_K(P_n)$ (where $K$ is a field of characteristic zero). Obviously, one can reformulate the Jacobian conjecture as the question of whether the Jacobian monoid $\Sigma(P_n) := \{\sigma \in \mathrm{End}_{K-alg}(P_n) \,|\, \mathcal{J}(\sigma) = 1\}$ is a *group*? The analogous Jacobian monoid $\Sigma = \Sigma(\Lambda_n)$ for the Grassmann algebra $\Lambda_n$ *is*, by a trivial reason, a *group*, it is a subgroup of the group $\mathrm{Aut}_K(\Lambda_n)$ of automorphisms of the Grassman algebra $\Lambda_n$. It turns out that properties of the Jacobian map $\mathcal{J}$ are closely related to properties of the Jacobian group $\Sigma$ which should be treated as the 'kernel' of the Jacobian map $\mathcal{J}$ despite the fact that $\mathcal{J}$ is *not* a homomorphism.

For a polynomial algebra $P_n = K[x_1, \dots, x_n]$, $n \geq 2$, over a field of characteristic zero $K$, the group $\mathrm{Aut}_K(P_n)$ of algebra automorphisms is an infinite algebraic group. We know little about this group for $n \geq 3$. There are three old open questions about the group $\mathrm{Aut}_K(P_n)$.

*Question 1. What are the defining relations of the algebraic group* $\mathrm{Aut}_K(P_n)$ *(as an infinite dimensional algebraic variety)?*

*Question 2. What are generators for* $\mathrm{Aut}_K(P_n)$*?*

*Question 3. What is a 'minimal' set of generators for* $\mathrm{Aut}_K(P_n)$*?*

The Jacobian Conjecture (if true) gives an answer to the first question. For the last two questions there are no even reasonable conjectures. In this paper, answers for 'analogous' questions are given for the Grassmann algebras.

It turns out that the Jacobian group $\Sigma$ is a large subgroup of $\mathrm{Aut}_K(\Lambda_n)$, so we start the paper considering the structure of the group $\mathrm{Aut}_K(\Lambda_n)$ and its subgroups. The Jacobian map and the Jacobian group are not transparent objects to deal with. Therefore, several (important) subgroups of $\mathrm{Aut}_K(\Lambda_n)$ are studied first. Some of them are given by explicit generators, another are defined via certain 'geometric' properties. That is why we study these subgroups in detail. They are building blocks in understanding the structure of the Jacobian map and the Jacobian group. Let us describe main results of the paper.

In the Introduction, $K$ is a *reduced commutative* ring with $\frac{1}{2} \in K$, $n \geq 2$ (though many results of the paper are true under milder assumptions, see in the text), $\Lambda_n = \Lambda_n(K) = K\lfloor x_1, \dots, x_n \rfloor$ be the Grassmann $K$-algebra and $\mathfrak{m} := (x_1, \dots, x_n)$ be its augmentation ideal. The algebra $\Lambda_n$ is endowed with the $\mathbb{Z}$-grading $\Lambda_n = \oplus_{i=0}^n \Lambda_{n,i}$ and $\mathbb{Z}_2$-grading $\Lambda_n = \Lambda_n^{\mathrm{ev}} \oplus \Lambda_n^{\mathrm{od}}$, and so each element $a \in \Lambda_n$ is a unique sum $a = a^{\mathrm{ev}} + a^{\mathrm{od}}$ where $a^{\mathrm{ev}} \in \Lambda_n^{\mathrm{ev}}$ and $a^{\mathrm{od}} \in \Lambda_n^{\mathrm{od}}$. For each $s \geq 2$, the algebra $\Lambda_n$ is also a $\mathbb{Z}_s$-graded algebra ($\mathbb{Z}_s := \mathbb{Z}/s\mathbb{Z}$).

**The structure of the group of automorphisms of the Grassmann algebra and its subgroups.** — In Sections 2 and 9, we study the group $G := \mathrm{Aut}_K(\Lambda_n(K))$ of $K$-algebra automorphisms of $\Lambda_n$ and various its subgroups (and their relations):

- $G_{gr}$, the subgroup of $G$ elements of which respect $\mathbb{Z}$-grading,
- $G_{\mathbb{Z}_2-gr}$, the subgroup of $G$ elements of which respect $\mathbb{Z}_2$-grading,
- $G_{\mathbb{Z}_s-gr}$, the subgroup of $G$ elements of which respect $\mathbb{Z}_s$-grading,

- $U := \{\sigma \in G \,|\, \sigma(x_i) = x_i + \cdots$ for all $i\}$ where the three dots mean bigger terms with respect to the $\mathbb{Z}$-grading,
- $G^{\rm od} := \{\sigma \in G \,|\, \sigma(x_i) \in \Lambda_{n,1} + \Lambda_n^{\rm od}$ for all $i\}$ and $G^{\rm ev} := \{\sigma \in G \,|\, \sigma(x_i) \in \Lambda_{n,1} + \Lambda_n^{\rm ev}$ for all $i\}$,
- $\mathrm{Inn}(\Lambda_n) := \{\omega_u : x \mapsto uxu^{-1}\}$ and $\mathrm{Out}(\Lambda_n) := G/\mathrm{Inn}(\Lambda_n)$, the groups of inner and outer automorphisms,
- $\Omega := \{\omega_{1+a} \,|\, a \in \Lambda_n^{\rm od}\}$,
- For each odd number $s$ such that $1 \leq s \leq n$, $\Omega(s) := \{\omega_{1+a} \,|\, a \in \sum_{1\leq j \text{ is odd}} \Lambda_{n,js}\}$,
- $\Gamma := \{\gamma_b \,|\, \gamma_b(x_i) = x_i + b_i,\ b_i \in \Lambda_n^{\rm od} \cap \mathfrak{m}^3, i = 1, \dots, n\}$, $b = (b_1, \dots, b_n)$,
- For each even number $s$ such that $3 \leq s \leq n$, $\Gamma(s) := \{\gamma_b \,|\, \text{all } b_i \in \sum_{j\geq 1} \Lambda_{n,1+js}\}$,
- $U^n := \{\tau_\lambda \,|\, \tau_\lambda(x_i) = x_i + \lambda_i x_1 \cdots x_n,\ \lambda = (\lambda_1, \dots, \lambda_n) \in K^n\} \simeq K^n$, $\tau_\lambda \leftrightarrow \lambda$,
- $\mathrm{GL}_n(K)^{op} := \{\sigma_A \,|\, \sigma_A(x_i) = \sum_{j=1}^n a_{ij}x_j,\ A = (a_{ij}) \in \mathrm{GL}_n(K)\}$,
- $\Phi := \{\sigma : x_i \mapsto x_i(1 + a_i) \,|\, a_i \in \Lambda_n^{\rm ev} \cap \mathfrak{m}^2,\ i = 1, \dots, n\}$.

If $K = \mathbb{C}$ the group $\Gamma\mathrm{GL}_n(\mathbb{C})^{op}$ was considered in [2]. If $K = k$ is a field of characteristic $\neq 2$ it was proved in [4] that $G$ is a semidirect product $\mathrm{Inn}(\Lambda_k(k)) \rtimes G_{\mathbb{Z}_2-gr}$. One can find a lot of information about the Grassmann algebra (i.e. the exterior algebra) in [3].

- (Lemma 2.8.(5)) $\Omega$ *is an abelian group canonically isomorphic to the additive group* $\Lambda_n^{\rm od}/\Lambda_n^{\rm od} \cap Kx_1 \cdots x_n$ via $\omega_{1+a} \mapsto a$.
- (Lemma 2.9, Corollary 2.15.(3)) $\mathrm{Inn}(\Lambda_n) = \Omega$ *and* $\mathrm{Out}(\Lambda_n) \simeq G_{\mathbb{Z}_2-gr}$.
- (Theorem 2.14) $U = \Omega \rtimes \Gamma$.
- (Theorem 2.17) $\Omega$ *is a maximal abelian subgroup of* $U$ *if* $n$ *is even* ($\Omega \supseteq U^n$)*; and* $\Omega U^n = \Omega \times U^n$ *is a maximal abelian subgroup of* $U$ *if* $n$ *is odd* ($\Omega \cap U^n = \{e\}$).
- (Theorem 2.14, Corollary 2.15, Lemma 2.16)
  1. $G = U \rtimes \mathrm{GL}_n(K)^{op} = (\Omega \rtimes \Gamma) \rtimes \mathrm{GL}_n(K)^{op}$,
  2. $G = \Omega \rtimes G_{\mathbb{Z}_2-gr}$, *and*
  3. $G = G^{\rm ev}G^{\rm od} = G^{\rm od}G^{\rm ev}$.
- (Lemma 2.16.(1)) $G^{\rm od} = G_{\mathbb{Z}_2-gr} = \Gamma \rtimes \mathrm{GL}_n(K)^{op}$.
- (Lemma 3.6) *Let* $s = 2, \dots, n$. *Then*

$$G_{\mathbb{Z}_s-gr} = \begin{cases} \Gamma(s) \rtimes \mathrm{GL}_n(K)^{op}, & \text{if } s \text{ is even,} \\ \Omega(s) \rtimes \mathrm{GL}_n(K)^{op}, & \text{if } s \text{ is odd.} \end{cases}$$

**The Jacobian matrix and an analogue of the Jacobian Conjecture for $\Lambda_n$.** — The even subalgebra $\Lambda_n^{\rm ev}$ of $\Lambda_n$ belongs to the centre of the algebra $\Lambda_n$. A $K$-linear map $\delta : \Lambda_n \to \Lambda_n$ is called a *left skew derivation* if $\delta(a_ia_j) = \delta(a_i)a_j + (-1)^i a_i\delta(a_j)$ for all homogeneous elements $a_i$ and $a_j$ of graded degree $i$ and $j$