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Abstract. — We study families of objects in Fukaya categories, specifically ones whose deformation behaviour is prescribed by the choice of an odd degree cohomology class. This leads to invariants of symplectic manifolds, which we apply to blowups along symplectic mapping tori.

Résumé **(Analogues abstraits du flux comme invariants des variétés symplectiques)**

Nous étudions des familles d'objets dans des catégories de Fukaya, en particulier celles dont le comportement infinitésimal est determiné par une classe de cohomologie de degré impair. Cette étude aboutit à des invariants des variétés symplectiques ; nous en tirons des conséquences pour les éclatements de tores d'applications symplectiques.

CONTENTS

INTRODUCTION

Motivation

An interesting invariant of a closed symplectic manifold M is its flux group, a subgroup of $H^1(M;\mathbb{R})$ obtained from the topology of loops of symplectic automorphisms [**74**, Section 10.2]. This can be effectively studied using Floer cohomology, one of the notable insights being that the flux group is always discrete [**80**]. Now consider the following question:

> *Are there flux-type subgroups in $H^{2k-1}(M;\mathbb{R})$, for $k > 1$, which can be nontrivial for manifolds with $H^1(M;\mathbb{R}) = 0$?*

The last clause excludes one obvious direction, which is to take the subgroup formed by the image of $[\omega_M^k]$ under all the maps $H^{2k}(M;\mathbb{R}) \to H^{2k-1}(M;\mathbb{R})$ induced by loops of symplectic automorphisms (this reproduces the flux group for $k = 1$, but it vanishes if $H^1(M;\mathbb{R}) = 0$, by the rigidity theorem [**61**]). Really, what the question is aiming for is a formalism in which higher degree differential forms replace the closed 1-forms in their usual relation to symplectic vector fields, so anything related to symplectic automorphism groups can't really be the answer. This clarification may make the whole endeavour seem quixotic.

Still, if one looks at it from the point of view of quantum cohomology $QH^*(M)$, the situation is less clear-cut. Passage to quantum cohomology generally reduces the grading to $\mathbb{Z}/2$, putting all odd degree cohomology formally on an equal footing (but degree one classes retain a more direct connection to geometry, because the quantum product with such a class remains equal to the classical cup product; this is by a version of the divisor axiom). In that vein, it turns out that one can give a partially positive answer to the question above, at least if one is willing to settle for an invariant which is somewhat more obscure, lacking the simplicity and geometric elegance of the flux group.

The examples

As an application, we consider a particular pair of 28-dimensional simply-connected symplectic manifolds (the following is only an outline of the construction, omitting many details and assumptions). Let K be a $K3$ surface, and $T \subset K$ a symplectically embedded two-torus. Take K^7, the product of seven copies of K, and blow up the 12-dimensional submanifold $T^2 \times K^2$. Denote the outcome by B^{triv}. This has a more interesting cousin B, defined in the same way but where the blowup locus is a product of T and the symplectic mapping torus of a certain automorphism of $K \times K$ (embedded into K^7 by using the h-principle).

It is known that the symplectic automorphism group of K has many connected components which are not detected by classical topological means (see for instance [**94**], or [**97**] for a mirror symmetry viewpoint). Based on that, one can ensure that B^{triv} is diffeomorphic to B, and that their symplectic structures are deformation equivalent. Nevertheless, for a specific choice of automorphism, we will show:

B and B^{triv} are not symplectically isomorphic.

The construction of these manifolds is similar to that of the first known examples of distinct but deformation equivalent symplectic structures [**73**], which were also based on blowing up. That paper used (roughly speaking) a bordism-valued refinement of Gromov-Witten theory as an invariant. Because such refinements are hard to define and compute, we can't say how they would behave in our situation. In any case, the approach taken in this paper is quite different.

The invariant. — Let's temporarily go back to the simpler case of symplectic mapping tori. The symplectic mapping torus of an automorphism f is a symplectic fibration over T which has trivial monodromy in one direction, and monodromy f in the other direction. Let's say for concreteness that $T = \mathbb{R}^2/\mathbb{Z}^2$ has coordinates (p, q), and that the monodromy is trivial in q-direction, and f in p-direction. The symplectic mapping torus contains plenty of Lagrangian submanifolds fibered over trivial circles $\{p\} \times S^1$. If one then moves such a Lagrangian submanifold by the time-one map of the symplectic vector field ∂_p, the effect is the same as applying f fibrewise. For suitable examples of f, this allows one to show that $[\mathrm{d}q]$ does not lie in the flux group, which distinguishes the mapping torus from the trivial one (a similar approach was used in [**92**]).

To make a more abstract version of the argument, we consider families of objects in the Fukaya category whose deformation is driven by the class $[\mathrm{d}q]$. The idea of introducing families of Lagrangian submanifolds into Floer cohomology theory is due to Fukaya [**33**], [**30**]. Generally speaking, it shows much potential for leading to fundamental insights as well as applications, but also encounters considerable foundational difficulties. Here, we bypass these issues by choosing a more constrained version of