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CRITICAL FUNCTIONAL FRAMEWORK AND MAXIMAL REGULARITY IN ACTION ON SYSTEMS OF INCOMPRESSIBLE FLOWS

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CRITICAL FUNCTIONAL FRAMEWORK AND MAXIMAL REGULARITY IN ACTION ON SYSTEMS OF INCOMPRESSIBLE FLOWS

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***Abstract*. —** This memoir is devoted to endpoint maximal regularity in Besov spaces for the evolutionary Stokes system in bounded or exterior domains of $\mathbb{R}^n$. We strive for *time independent* a priori estimates with L_1 time integrability.

In the whole space case, endpoint maximal regularity estimates are well known and have proved to be spectacularly powerful to investigate the well-posedness issue of PDEs related to fluid mechanics. They have been extended recently by the authors to the half-space setting [**15**]. The present work deals with the bounded and exterior domain cases. Although in both situations the Stokes system may be localized and reduced up to low order terms to the half-space and whole space cases, the exterior domain case is more involved owing to a bad control on the low frequencies of the solution (no Poincaré inequality is available whatsoever). In order to glean some global-in-time integrability, we adapt to the Besov space setting the approach introduced by P. Maremonti and V.A. Solonnikov in [**39**]. The price to pay is that we end up with estimates in intersections of Besov spaces, rather than in a single Besov space.

As a first application of our work, we solve locally for large data or globally for small data, the (slightly) inhomogeneous incompressible Navier-Stokes equations in critical Besov spaces, in an exterior domain. After observing that the L_1 time integrability allows to determine globally the streamlines of the flow, the whole system is recast in the Lagrangian coordinates setting. This, in particular, enables us to consider discontinuous densities, as in [**17**], [**19**].

The second application concerns a low Mach number system that has been studied recently in the whole space setting by the first author and X. Liao [**14**].

***Résumé* (Régularité critique, régularité maximale et application à la mécanique des fluides incompressibles)**

Ce mémoire traite de la régularité maximale limite dans les espaces de Besov pour le système de Stokes non stationnaire en domaine borné ou extérieur. Nous avons en vue des estimations avec intégrabilité globale en temps.

Les inégalités de régularité maximale limite sont bien connues dans l'espace entier et ont joué un rôle spectaculaire dans l'étude du problème de Cauchy associé à diverses EDPs de la mécanique des fluides. Ces inégalités ont été adaptées récemment par les auteurs au cas du demi-espace [**15**]. Nous considérons ici des domaines bornés ou extérieurs. Bien que dans les deux situations le système de Stokes puisse être localisé et l'étude, ramenée à celle de l'espace entier ou du demi-espace, le cas du domaine extérieur est plus compliqué car on ne dispose pas de contrôle *a priori* sur les basses fréquences via une inégalité de Poincaré par exemple. Afin d'exhiber une certaine forme d'intégrabilité globale en temps, nous adaptons au cadre Besov le travail de P. Maremonti et V.A. Solonnikov [**39**]. Nous obtenons ainsi le type d'inégalités voulu, mais dans l'intersection d'espaces de Besov.

Comme première application de ces nouvelles inégalités, nous résolvons localement (données initiales grandes) ou globalement (données initiales petites) les équations de Navier-Stokes incompressibles faiblement non homogènes en domaine extérieur dans des espaces de Besov critiques. La propriété d'intégrabilité L^1 en temps à valeurs Lipschitz pour le champ de vitesses solution assure l'équivalence entre les formulations lagrangiennes et eulériennes du système. Passer en coordonnées lagrangiennes permet de considérer des données initiales avec densité discontinue, comme observé récemment dans [**17**], [**19**].

Comme deuxième application, nous résolvons un système limite qui apparaît dans le régime à faible nombre de Mach et a été étudié récemment par le premier auteur et X. Liao [**14**].

CONTENTS

CHAPTER 1

INTRODUCTION

Description of motion of Newtonian fluids is based on the physical and thermodynamical laws governing the conservation of momentum, energy and mass. We expect in general information concerning these quantities to be enough to find out the velocity field at each point of the fluid region and, at least, on some time interval $[0, T]$ if the initial time is $t = 0$. This is the Eulerian description of the fluid.

Another fundamental physical information is the knowledge of the *streamlines* or *particle paths* corresponding to the evolution of infinitesimal particles or fluid parcels. It is given by the following Ordinary Differential Equation:

$$\frac{dX}{dt} = v(t, X), \qquad X|_{t=0} = y. \tag{1.1}$$

Here y is the initial position of a particle of the fluid and $X(t, y)$ denotes the position of that particle at time t under the action of the velocity field v. Knowing X thus gives the evolution of an infinitesimal fluid parcel *labelled by its initial position* y as it moves through space and time. This is the Lagrangian description of the fluid under consideration. Equation (1.1) gives the relationship between the two descriptions of flows, that is the Eulerian and Lagrangian ones. The Eulerian coordinates system (t, x) uses the position x of the material at time t, while the Lagrangian coordinates system (t, y) uses the initial position y of a point of the medium. The change of coordinates is governed by the following identity which is the integrated counterpart of (1.1):

$$x = X(t, y) \quad \text{with} \quad X(t, y) = y + \int_0^t v(\tau, X(\tau, y))\, d\tau. \tag{1.2}$$

From the mathematical viewpoint, the basic question is whether those two descriptions are indeed equivalent: what are the conditions ensuring that one may go from one system of coordinates to the other without any loss of information on the flow ? From the basic theory of Ordinary Differential Equations, we know that, roughly, the minimal assumption is that

$$\nabla v \in L_1(0,T; L_{\infty,\mathrm{loc}}(\Omega)), \tag{1.3}$$

where Ω is the fluid domain. This in particular ensures (1.2) to have a unique solution X that is continuous in time and Lipschitz in space (see [**4**], [**23**], [**45**] for more general results concerning the flow and transport equations).

In the present work, we would like to find a functional framework – the largest one if possible – ensuring the velocity field to satisfy (1.3) and the system we are looking at, to be well-posed. We have in mind models describing the evolution of incompressible flows with nonconstant densities and, more specifically, mixtures of incompressible homogeneous fluids. Such models possess some invariance with respect to appropriate time and space dilations and it has been observed in many situations that the optimal functional framework – the so-called critical one – for studying the corresponding governing equations should have the same invariance (see the introduction of Chapter 5 for more explanations).

Resuming to the study of mixtures of incompressible flows, the basic question is whether the following initial configuration:

$$\rho_0 = 1 + \sigma\chi_A, \tag{1.4}$$

where σ denotes some constant and χ_A, the characteristic function of some subset A of Ω, is stable through the time evolution. According to the Lagrangian description introduced above, we expect the density to be transported by the velocity field and thus to read

$$\rho(t) = 1 + \sigma\chi_{A(t)} \quad \text{with} \quad A(t) := X(t, A). \tag{1.5}$$

Note that if (1.3) is fulfilled then the flow X is Lipschitz and thus $A(t)$ remains Lipschitz during the evolution, if it is Lipschitz initially.

To make it more concrete, consider the following *inhomogeneous incompressible Navier-Stokes system*:

$$\begin{cases} \rho_t + u\cdot\nabla\rho = 0 & \text{in} \quad (0,T)\times\Omega, \\ \rho(u_t + u\cdot\nabla u) - \nu\Delta u + \nabla P = 0 & \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in} \quad (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{at} \quad (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0, \qquad \rho|_{t=0} = \rho_0 & \text{on} \quad \Omega. \end{cases} \tag{1.6}$$