

# Mémoires

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

GLOBAL ASPECTS OF THE  
REDUCIBILITY OF  
QUASIPERIODIC COCYCLES  
IN SEMISIMPLE COMPACT  
LIE GROUPS

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# GLOBAL ASPECTS OF THE REDUCIBILITY OF QUASIPERIODIC COCYCLES IN SEMISIMPLE COMPACT LIE GROUPS

Nikolaos Karaliolios

**Abstract.** — In this mémoire we study quasiperiodic cocycles in semi-simple compact Lie groups. For the greatest part of our study, we will focus ourselves to one-frequency cocycles. We will prove that  $C^\infty$ -reducible cocycles are dense in the  $C^\infty$  topology, for a full measure set of frequencies. Moreover, we will show that every cocycle (or an appropriate iterate of it, if homotopy appears as an obstruction) is almost torus-reducible (i.e. can be conjugated arbitrarily close to cocycles taking values in an abelian subgroup of  $G$ ). In the course of the proof we will firstly define two invariants of the dynamics, which we will call *energy* and *degree* and which give a preliminary distinction between (almost-)reducible and non-reducible cocycles. We will then take up the proof of the density theorem. We will show that an algorithm of *renormalization* converges to perturbations of simple models, indexed by the degree. Finally, we will analyze these perturbations using methods inspired by K.A.M. theory.

**Résumé.** — Ce mémoire porte sur l'étude des cocycles quasi-périodiques à valeurs dans des groupes de Lie compacts semi-simples. Nous nous restreindrons au cas des cocycles à une fréquence. Nous démontrons que, pour un ensemble de fréquences de mesure de Lebesgue pleine, l'ensemble des cocycles  $C^\infty$  qui sont  $C^\infty$ -réductibles sont  $C^\infty$ -denses. De plus, sous la même condition arithmétique, nous démontrons que tout cocycle (quitte à l'itérer afin de simplifier suffisamment l'homotopie du lacet dans le groupe), est presque tore-réductible (c'est-à-dire qu'il peut être conjugué arbitrairement proche à des cocycles prenant valeurs dans un sous-groupe abélien donné de  $G$ ).

Le premier pas de la démonstration est l'obtention de deux invariants de la dynamique, qu'on appelle *énergie* et *degré*, qui distinguent en particulier les cocycles (presque-)réductibles des cocycles non-réductibles. On entamera ensuite la démonstration du théorème principal. Nous démontrons dans un second temps qu'un algorithme dit de *renormalisation* permet de ramener l'étude de tout cocycle à celle des perturbations de modèles simples indexés par le degré. Nous analysons ensuite ces perturbations par des méthodes inspirées de la théorie K.A.M.

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# CHAPTER 1

## INTRODUCTION

This mémoire is concerned with the study of quasiperiodic cocycles in semisimple compact Lie groups. Cocycles are discrete dynamical systems whose phase space is a fibered space  $X \times E \rightarrow X$ . Fibered dynamics is given by the iteration of a mapping of the type

$$(T, f) : X \times E \longrightarrow X \times E, \quad (x, e) \longmapsto (Tx, f(x, e))$$

where  $T$  is a mapping of  $X$  into itself, and  $f : X \times E \rightarrow E$ . Consequently, the fiber  $\{x\} \times E$  is mapped into the fiber  $\{Tx\} \times E$  following  $e \mapsto f(x, e)$ . The notation  $\text{SW}(X, E)$  for the set of such dynamical systems is classical. If  $E$  is a group or a space on which a group acts, this kind of fibered dynamics is called a cocycle. We can then note a cocycle by

$$(T, f) : X \times E \longrightarrow X \times E, \quad (x, e) \longmapsto (Tx, f(x).e)$$

with  $f : X \rightarrow E$  and the dot  $.$  stands for the group multiplication or action.

The  $n$ -th iterate of the cocycle  $(T, f)$ ,  $n \geq 1$ , is of the form

$$(T, f)^n.(x, e) = (T^n x, f(T^{n-1}x) \circ \cdots \circ f(x).e)$$

We say that two cocycles  $\psi_i = (T, f_i) \in \text{SW}(X, E)$ ,  $i = 1, 2$ , over the same transformation are (semi-)conjugate iff there exists  $g : X \rightarrow E$  such that

$$\psi_1 \circ (\text{Id}, g) = (\text{Id}, g) \circ \psi_2$$

and we remark that it is a notion stronger than that of dynamical (semi-)conjugation by a mapping  $h : X \times E \rightarrow X \times E$  satisfying  $\psi_1 \circ h = h \circ \psi_2$ , since conjugation of cocycles preserves the fibered-space structure of  $X \times E$ .

In general contexts we suppose that  $(X, \mu)$ , the *basis* of the dynamics, is a measured space and that  $T$  is ergodic with respect to  $\mu$ . A particular case, which brings us to the subject of our study, occurs when  $X = \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ ,

a  $d$ -dimensional torus, and  $T = R_\alpha : x \mapsto x + \alpha$ , is a minimal translation, and therefore uniquely ergodic with respect to the Haar measure on the torus. These cocycles are called *quasiperiodic* and  $\alpha$  is called the *frequency* of the cocycle.

We remark here that, depending on the structure of  $E$ , the *fibers*, we can define measurable cocycles, or  $C^k$ -differentiable cocycles, with  $k \in \mathbb{N} \cup \{\infty, \omega\}$  ( $\omega$  stands for real analytic), according to the regularity of the mapping  $f$ .

Cocycles in linear groups come up naturally in dynamical systems. For example, if  $\varphi$  is a diffeomorphism of the torus  $\mathbb{T}^d$ , its differential defines a cocycle on  $T\mathbb{T}^d \approx \mathbb{T}^d \times \mathbb{R}^d$  in a natural way by

$$(x, y) \longmapsto (\varphi(x), D\varphi(x).y)$$

An other class of examples, closer to our subject, is that of fibered linear flows. Such a flow is defined as the fundamental solution of the system of ODEs

$$X' = F(\theta).X, \quad \theta' = \omega = (\alpha, 1) \in \mathbb{T}^{d+1}$$

where  $F : \mathbb{T}^{d+1} \rightarrow g$ , and  $g$  is a matrix algebra in  $M_N(\mathbb{R})$ . The map of first return in the vertical circle  $\mathbb{T} \hookrightarrow \mathbb{T}^d \times \mathbb{T}$  is a cocycle on  $\{0\} \times \mathbb{T}^d \times \mathbb{R}^N$ . It is quasiperiodic if  $\{\mathbb{R}\omega \bmod \mathbb{Z}^{d+1}\}$  is dense in  $\mathbb{T}^{d+1}$ .

When  $\alpha$  is rational, such flows are perfectly understood thanks to the Floquet representation of solutions: The solutions of the system of ODEs

$$(1) \quad X' = U(\theta).X, \quad \theta' = k \in \mathbb{Q}^d$$

are of the form  $X(t) = B(kt + \theta_0).e^{tU_0(\theta_0)}.X_0$  where  $B(\cdot)$  is a  $2\mathbb{Z}^d$ -periodic map in the matrix group of algebra  $g$ . For cocycles, the Floquet representation of a solution corresponds to conjugation of a cocycle to a constant one: a cocycle  $(R_\alpha, f)$  is called constant if  $f : \mathbb{T}^d \rightarrow G$  is a constant mapping. Such cocycles are called *reducible*.

This normal form theorem breaks down when  $\alpha$  is irrational, and the goal of the theory is to examine the density properties of Floquet-type solutions, and the possibility of approximation of any given vector field with a field admitting Floquet-type solutions. We now restrict ourselves to the case of cocycles in  $\mathbb{T}^d \times G$ , where  $G$  is a semisimple compact Lie group, such as  $SU(N)$  or  $SO(N)$ .

The first step is the study of vector fields close to constants, or cocycles close to constants. In this setting the breakdown of Floquet theory is attributed to *small divisor* phenomena. The use of K.A.M. machinery allows, however, to give an affirmative answer to both questions, for a full measure set of frequencies (called Diophantine) (see [29] and the references therein). Moreover, Floquet representations occur in full measure sets for generic one-parameter