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GLOBAL ASPECTS OF THE REDUCIBILITY OF QUASIPERIODIC COCYCLES IN SEMISIMPLE COMPACT LIE GROUPS

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GLOBAL ASPECTS OF THE REDUCIBILITY OF QUASIPERIODIC COCYCLES IN SEMISIMPLE COMPACT LIE GROUPS

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***Abstract.* —** In this mémoire we study quasiperiodic cocycles in semi-simple compact Lie groups. For the greatest part of our study, we will focus ourselves to one-frequency cocyles. We will prove that C^{∞}-reducible cocycles are dense in the C^{∞} topology, for a full measure set of frequencies. Moreover, we will show that every cocycle (or an appropriate iterate of it, if homotopy appears as an obstruction) is almost torus-reducible (i.e. can be conjugated arbitrarily close to cocycles taking values in an abelian subgroup of G). In the course of the proof we will firstly define two invariants of the dynamics, which we will call *energy* and *degree* and which give a preliminary distinction between (almost-)reducible and non-reducible cocycles. We will then take up the proof of the density theorem. We will show that an algorithm of *renormalization* converges to perturbations of simple models, indexed by the degree. Finally, we will analyze these perturbations using methods inspired by K.A.M. theory.

Résumé. — Ce mémoire porte sur l'étude des cocycles quasi-périodiques à valeurs dans des groupes de Lie compacts semi-simples. Nous nous restreindrons au cas des cocycles à une fréquence. Nous démontrons que, pour un ensemble de fréquences de mesure de Lebesgue pleine, l'ensemble des cocycles C^∞ qui sont C^∞-réductibles sont C^∞-denses. De plus, sous la même condition arithmétique, nous démontrons que tout cocycle (quitte à l'itérer afin de simplifier suffisamment l'homotopie du lacet dans le groupe), est presque tore-réductible (c'est-à-dire qu'il peut être conjugué arbitrairement proche à des cocycles prenant valeurs dans un sous-groupe abélien donné de G).

Le premier pas de la démonstration est l'obtention de deux invariants de la dynamique, qu'on appelle *énergie* et *degré*, qui distinguent en particulier les cocycles (presque-)réductibles des cocycles non-réductibles. On entamera ensuite la démonstration du théorème principal. Nous démontrons dans un second temps qu'un algorithme dit de *renormalisation* permet de ramener l'étude de tout cocycle à celle des perturbations de modèles simples indexés par le degré. Nous analysons ensuite ces perturbations par des méthodes inspirées de la théorie K.A.M.

CONTENTS

CHAPTER 1

INTRODUCTION

This mémoire is concerned with the study of quasiperiodic cocycles in semisimple compact Lie groups. Cocycles are discrete dynamical systems whose phase space is a fibered space $X \times E \to X$. Fibered dynamics is given by the iteration of a mapping of the type

$$(T, f) : X \times E \longrightarrow X \times E, \quad (x, e) \longmapsto (Tx, f(x, e))$$

where T is a mapping of X into itself, and $f : X \times E \to E$. Consequently, the fiber $\{x\} \times E$ is mapped into the fiber $\{Tx\} \times E$ following $e \mapsto f(x, e)$. The notation $\mathrm{SW}(X, E)$ for the set of such dynamical systems is classical. If E is a group or a space on which a group acts, this kind of fibered dynamics is called a cocycle. We can then note a cocycle by

$$(T, f) : X \times E \longrightarrow X \times E, \quad (x, e) \longmapsto (Tx, f(x).e)$$

with $f : X \to E$ and the dot . stands for the group multiplication or action.

The n-th iterate of the cocycle (T, f), $n \geq 1$, is of the form

$$(T, f)^n.(x, e) = (T^n x, f(T^{n-1}x) \circ \cdots \circ f(x).e)$$

We say that two cocycles $\psi_i = (T, f_i) \in \mathrm{SW}(X, E)$, $i = 1, 2$, over the same transformation are (semi-)*conjugate* iff there exists $g : X \to E$ such that

$$\psi_1 \circ (\mathrm{Id}, g) = (\mathrm{Id}, g) \circ \psi_2$$

and we remark that it is a notion stronger than that of dynamical (semi-)conjugation by a mapping $h : X \times E \circlearrowleft$ satisfying $\psi_1 \circ h = h \circ \psi_2$, since conjugation of cocycles preserves the fibered-space structure of $X \times E$.

In general contexts we suppose that (X, μ), the *basis* of the dynamics, is a measured space and that T is ergodic with respect to μ. A particular case, which brings us to the subject of our study, occurs when $X = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, a d-dimensionan torus, and $T = R_\alpha : x \mapsto x + \alpha$, is a minimal translation, and therefore uniquely ergodic with respect to the Haar measure on the torus. These cocycles are called *quasiperiodic* and α is called the *frequency* of the cocycle.

We remark here that, depending on the structure of E, the *fibers*, we can define measurable cocycles, or C^k-differentiable cocycles, with $k \in \mathbb{N} \cup \{\infty, \omega\}$ (ω stands for real analytic), according to the regularity of the mapping f.

Cocycles in linear groups come up naturally in dynamical systems. For example, if φ is a diffeomorphism of the torus $\mathbb{T}^d$, its differential defines a cocycle on $T\,\mathbb{T}^d \approx \mathbb{T}^d \times \mathbb{R}^d$ in a natural way by

$$(x, y) \longmapsto (\varphi(x), D\varphi(x).y)$$

An other class of examples, closer to our subject, is that of fibered linear flows. Such a flow is defined as the fundamental solution of the system of ODEs

$$X' = F(\theta).X, \quad \theta' = \omega = (\alpha, 1) \in \mathbb{T}^{d+1}$$

where $F : \mathbb{T}^{d+1} \to g$, and g is a matrix algebra in $M_N(\mathbb{R})$. The map of first return in the vertical circle $\mathbb{T} \hookrightarrow \mathbb{T}^d \times \mathbb{T}$ is a cocycle on $\{0\} \times \mathbb{T}^d \times \mathbb{R}^N$. It is quasiperiodic if $\{\mathbb{R}\omega \mod \mathbb{Z}^{d+1}\}$ is dense in $\mathbb{T}^{d+1}$.

When α is rational, such flows are perfectly understood thanks to the Floquet representation of solutions: The solutions of the system of ODEs

$$X' = \mathrm{U}(\theta).X, \quad \theta' = k \in \mathbb{Q}^d \tag{1}$$

are of the form $X(t) = B(kt+\theta_0).e^{tU_0(\theta_0)}.X_0$ where $B(\cdot)$ is a $2\mathbb{Z}^d$-periodic map in the matrix group of algebra g. For cocycles, the Floquet representation of a solution corresponds to conjugation of a cocycle to a constant one: a cocycle (R_α, f) is called constant if $f : \mathbb{T}^d \to G$ is a constant mapping. Such cocycles are called *reducible*.

This normal form theorem breaks down when α is irrational, and the goal of the theory is to examine the density properties of Floquet-type solutions, and the possibility of approximation of any given vector field with a field admitting Floquet-type solutions. We now restrict ourselves to the case of cocycles in $\mathbb{T}^d \times G$, where G is a semisimple compact Lie group, such as $\mathrm{SU}(N)$ or $\mathrm{SO}(N)$.

The first step is the study of vector fields close to constants, or cocycles close to constants. In this setting the breakdown of Floquet theory is attributed to *small divisor* phenomena. The use of K.A.M. machinery allows, however, to give an affirmative answer to both questions, for a full measure set of frequencies (called Diophantine) (see [**29**] and the references therein). Moreover, Floquet representations occur in full measure sets for generic one-parameter