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ROLLING OF MANIFOLDS AND CONTROLLABILITY IN DIMENSION THREE

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ROLLING OF MANIFOLDS AND CONTROLLABILITY IN DIMENSION THREE

Yacine Chitour, Petri Kokkonen

Abstract. — We present the rolling (or development) of one smooth connected complete Riemannian manifold (M, g) onto another one $(\widehat{M}, \widehat{g})$ of equal dimension $n \geq 2$ where there is no relative spin or slip of one manifold with respect to the other one. Relying on geometric control theory, we provide an intrinsic description of the two constraints "without spinning" and "without slipping" in terms of the Levi-Civita connections ∇^g and $\nabla^{\widehat{g}}$ by defining corresponding vector fields distributions in the appropriate state space. We then address the issue of complete controllability for that rolling problem. We first establish basic global properties for the reachable set and investigate the associated Lie bracket structure. In particular, we point out the role played by a curvature tensor defined on the state space, that we call the *rolling curvature*. When the two manifolds are three-dimensional, we give a complete local characterization of the reachable sets and, in particular, we identify necessary and sufficient conditions for the existence of a non open orbit. In addition to the trivial case where the manifolds (M, g) and $(\widehat{M}, \widehat{g})$ are (locally) isometric, we show that (local) non controllability occurs if and only if (M, g) and $(\widehat{M}, \widehat{g})$ are either warped products or contact manifolds with additional restrictions that we precisely describe.

***Résumé* (Roulement de variétés et commandabilité en dimension trois)**

Nous présentons le roulement (ou développement) d'une variété riemannienne connexe (M,g) sur une autre $(\widehat{M},\widehat{g})$ de dimension égale $n\geq 2$, lorsqu'il y a pas de glissement ni spin de l'une par rapport à l'autre. Nous donnons une description intrinsèque des contraintes « sans glissement » et « sans spin » à l'aide des connections de Levi-Civita ∇^g and $\nabla^{\widehat{g}}$ afin de définir la distribution associée à (R) dans l'espace d'état approprié. Nous donnons les premières propriétés globales pour les ensembles atteignables et nous étudions la structure d'algèbre de Lie correspondante. En particulier, nous caractérisons le rôle crucial joué par un tensor courbure dans l'espace d'état que nous appelons *courbure de roulement*. Lorsque les deux variétés sont de dimension 3, nous donnons une caractérisation complète de la structure locale des ensembles atteignables et en particulier celles des orbites non ouvertes. En plus du cas trivial où les variétés (M,g) et $(\widehat{M},\widehat{g})$ sont (localement) isométriques, nous montrons que la non commandabilité locale a lieu si et seulement si (M,g) et $(\widehat{M},\widehat{g})$ sont des produits tordus ou des variétés de contact avec une description précise.

CONTENTS

CHAPTER 1

INTRODUCTION

We study the rolling of a manifold over another one. Unless otherwise precized, manifolds are smooth, connected, oriented, of finite dimension $n \geq 2$, endowed with a complete Riemannian metric. The rolling is assumed to be without spinning nor slipping and we refer to it as the rolling (R) since it is possible to have another rolling problem just assuming a no-slipping condition (*cf.* [**23**]).

When both manifolds are isometrically embedded into an Euclidean space, the rolling problem (R) is classical in differential geometry through the notions of "development of a manifold" and "rolling maps", see [**41**] and references therein.

To get an intuitive grasp of the problem, consider the rolling problem (R) of a 2D convex surface S_1 onto another one S_2 in the euclidean space $\mathbb{R}^3$. The most classical such example is the so-called *plate-ball* problem, *i.e.*, a sphere rolling onto a plane in $\mathbb{R}^3$, (*cf.* [**21**] and [**32**]). The two surfaces are in contact, *i.e.* they have a common tangent plane at the contact point and, equivalently, their exterior normal vectors are opposite at the contact point.

If $\gamma : [0, T] \to S_1$ is a C^1 regular curve on S_1, one says that S_1 rolls onto S_2 along γ without spinning nor slipping if the following holds. The curve traced on S_1 by the contact point is equal to γ and let $\widehat{\gamma} : [0, T] \to S_2$ be the curve traced on S_2 by the contact point. At every time $t \in [0, T]$ the relative orientation of S_2 with respect to S_1 is measured by the angle $\theta(t)$ between $\dot{\gamma}(t)$ and $\dot{\widehat{\gamma}}(t)$ in the common tangent plane at the contact point and let Q be the state space of the rolling problem (which is therefore five dimensional since a point in Q is defined by fixing a point on S_1, a point on S_2 and an angle in S^1, the unit circle). The no-slipping condition says that $\dot{\widehat{\gamma}}(t)$ is equal to $\dot{\gamma}(t)$ rotated by the angle $\theta(t)$ and the no-spinning condition characterizes $\dot{\theta}(t)$ in term of the surface elements at $\gamma(t)$ and $\widehat{\gamma}(t)$ respectively. Then, once a point on S_2 and an angle are chosen at time $t = 0$, the curves $\hat{\gamma}$ and θ are uniquely determined.

The most basic issue in geometric control theory linked to the rolling problem (R) is that of *controllability*, *i.e.* to determine, for two given points q_{init} and q_{final} in the state space Q, if there exists a curve γ so that the rolling of S_1 onto S_2 along γ steers the system from q_{init} to q_{final}. If this is the case for every points q_{init} and q_{final} in Q, then the rolling of S_1 onto S_2 is said to be *completely controllable*.

If the manifolds rolling on each other are two-dimensional, the controllability issue is well-understood thanks to the work of [**2**], [**7**] and [**27**] especially. For instance, in the simply connected case, the rolling (R) is completely controllable if and only if the manifolds are not isometric. In the case where the manifolds are isometric, [**2**] also provides a description of the reachable sets in terms of isometries between the manifolds.

In particular, these reachable sets are immersed submanifolds of Q of dimension either 2 or 5. In case the manifolds rolling on each other are isometric convex surfaces, [**27**] provides a beautiful description of a two dimensional reachable set: consider the initial configuration given by two (isometric) surfaces in contact so that one is the image of the other one by the symmetry with respect to the (common) tangent plane at the contact point. Then, this symmetry property (chirality) is preserved along the rolling (R). Note that if the (isometric) convex surfaces are not spheres nor planes, the reachable set starting at a contact point where the Gaussian curvatures are distinct, is open (and thus of dimension 5).

From a robotics point of view, once the controllability is well-understood, the next issue to address is that of *motion planning*, *i.e.*, defining an effective procedure that produces, for every pair of points $(q_{\text{init}}, q_{\text{final}})$ in the state space Q, a curve $\gamma_{q_{\text{init}}, q_{\text{final}}}$ so that the rolling of S_1 onto S_2 along $\gamma_{q_{\text{init}}, q_{\text{final}}}$ steers the system from q_{init} to q_{final}. In [**9**], an algorithm based on the continuation method was proposed to tackle the rolling problem (R) of a strictly convex compact surface onto an Euclidean plane. That algorithm was also proved in [**9**] to be convergent and it was numerically implemented in [**4**] (see also [**26**] for another algorithm).

The rolling problem (R) is traditionally presented by isometrically embedding the rolling manifolds M and $\widehat{M}$ in an Euclidean space (*cf.* [**33**], [**41**], [**19**]) since it is the most intuitive way to provide a rigorous meaning to the notions of relative spin (or twist) and relative slip of one manifold with respect