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COMPACTNESS PROPERTIES OF PERTURBED SUB-STOCHASTIC C_0-SEMIGROUPS ON $L^1(\mu)$ WITH APPLICATIONS TO DISCRETENESS AND SPECTRAL GAPS

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COMPACTNESS PROPERTIES OF PERTURBED SUB-STOCHASTIC C_0-SEMIGROUPS ON $L^1(\mu)$ WITH APPLICATIONS TO DISCRETENESS AND SPECTRAL GAPS

Mustapha Mokhtar-Kharroubi

***Abstract.* —** We deal with positive C_0-semigroups $(U(t))_{t\geqslant 0}$ of contractions in $L^1(\Omega;\mathcal{A},\mu)$ with generator T where $(\Omega;\mathcal{A},\mu)$ is an abstract measure space and provide a systematic approach of compactness properties of perturbed C_0-semigroups $\left(e^{t(\text{“}T-V\text{”})}\right)_{t\geq 0}$ (or their generators) induced by singular potentials $V:(\Omega;\mu)\to\mathbb{R}_+$. More precise results are given in metric measure spaces (Ω,d,μ). This new construction is based on several ingredients: new a priori estimates peculiar to L^1-spaces, local weak compactness assumptions on unperturbed operators, “Dunford-Pettis” arguments and the assumption that the sublevel sets $\Omega_M:=\{x;V(x)\leq M\}$ are “thin at infinity with respect to $(U(t))_{t\geqslant 0}$”. We show also how spectral gaps occur when the sublevel sets are not “thin at infinity”. This formalism combines intimately the kernel of $(U(t))_{t\geqslant 0}$ and the sublevel sets Ω_M. Indefinite potentials are also dealt with. Various applications to convolution semigroups, weighted Laplacians and Witten Laplacians on 1-forms are given.

***Résumé* (Propriétés de compacité de semigroupes sous-stochastiques perturbés dans L^1 et applications aux spectres discrets et aux trous spectraux)**

Nous traitons de C_0-semigroupes à contractions positifs $(U(t))_{t\geqslant 0}$ dans $L^1(\Omega;\mathcal{A},\mu)$ de générateur T où $(\Omega;\mathcal{A},\mu)$ est un espace mesuré abstrait et donnons une approche systématique des propriétés de compacité de C_0-semigroupes perturbés $\left(e^{t(\text{``}T-V\text{''})}\right)_{t\geq 0}$ (ou de leurs générateurs) induites par des potentiels singuliers $V:(\Omega;\mu)\to\mathbb{R}_+$. Des résultats plus précis sont donnés pour des espaces métriques mesurés (Ω,d,μ). Cette nouvelle construction repose sur plusieurs ingrédients : de nouvelles estimations a priori propres aux espaces L^1, des hypothèses de compacité locale faible sur les opérateurs non perturbés, des arguments de type « Dunford-Pettis » et l'hypothèse que les sous-ensembles de niveau $\Omega_M := \{x; V(x)\leq M\}$ sont « fins à l'infini par rapport à $(U(t))_{t\geqslant 0}$ ». Nous montrons aussi l'apparition de trous spectraux lorsque les sous-ensembles de niveaux Ω_M ne sont pas « fins à l'infini par rapport à $(U(t))_{t\geqslant 0}$ ». Ce formalisme combine intimement le noyau de $(U(t))_{t\geqslant 0}$ et les sous ensembles de niveau Ω_M. Les potentiels indéfinis sont aussi traités. Des applications variées aux semigroupes de convolution, aux Laplaciens à poids et aux Laplaciens de Witten sur les 1-formes sont données.

CONTENTS

CHAPTER 1

INTRODUCTION

This work is an improved version of [**44**] and provides new functional analytic tools and results on perturbation theory and spectral analysis of substochastic C_0-semigroups in L^1 spaces and also various related results of applied interest. Before outlining the content of this work, some information in Hilbert space setting is worth mentioning. According to a classical result going back at least to K. Friedrichs [**18**], the spectrum of a Schrödinger operator in $L^2(\mathbb{R}^N)$

$$(-\Delta) \dot{+} V \quad \text{(form-sum)}$$

is discrete (i.e. consists of isolated eigenvalues with finite multiplicity) or equivalently $(-\Delta) \dot{+} V$ has a compact resolvent for nonnegative potentials

$$V \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{such that} \quad \lim_{|x| \to \infty} V(x) = +\infty.$$

Of course, it is also known since a long time that this condition is not necessary since F. Rellich [**64**] already observed for example that for the potential

$$V(x_1, x_2) = x_1^2 x_2^2, \tag{1}$$

$(-\Delta) \dot{+} V$ is still resolvent compact in $L^2(\mathbb{R}^2)$ even if $V(x_1, x_2)$ fails to go to $+\infty$ at infinity near the axes. Besides K. Friedrichs [**18**], the literature on discreteness of the spectrum of Schrödinger operators goes back to A.M. Molchanov [**54**] and is now considerable; we refer to the survey [**69**] and also to the more recent paper [**41**] for more developments. This literature deals with Schrödinger operators on more general non-compact Riemannian manifolds and provides optimal (i.e. necessary and sufficient) conditions of discreteness in terms of Wiener capacity of suitable sets. Such sharp results are not always of simple practical use, but sufficient or necessary conditions in terms of measures are also available. For instance, we note A.M. Molchanov's necessary condition of discreteness

$$\int_{B(x,r)} V(y)\,\mathrm{d}y \longrightarrow +\infty \quad \text{as } |x| \to \infty$$

where $B(x,r)$ is the ball centered at x with radius r. We note also that if for any $M > 0$ the sublevel set

$$\Omega_M := \big\{y; V(y) \leq M\big\}$$

is "thin at infinity" in the sense that for some $r > 0$

$$\big|B(x,r) \cap \Omega_M\big| \longrightarrow 0 \ \text{ as } |x| \to \infty \tag{2}$$

(here $\big|\Xi\big|$ refers to Lebesgue measure of a measurable set Ξ) then $(-\Delta) \dotplus V$ has a discrete spectrum, see [**69**], Corollary 10.2, p. 268.

In ([**20**], Lemma 5 and Remark 2), it is observed that the sublevel sets of a nonnegative function V are "thin at infinity" if and only if for some $r > 0$

$$\int_{B(x,r)} \frac{1}{1+V(y)}\,\mathrm{d}y \longrightarrow 0 \quad \text{as } |x| \to \infty; \tag{3}$$

the argument relies on the simple double inequality (for arbitrary $M > 0$)

$$\frac{1}{1+M}\big|B(x,r) \cap \Omega_M\big| \leq \int_{B(x,r)} \frac{1}{1+V(y)}\,\mathrm{d}y,$$

$$\int_{B(x,r)} \frac{1}{1+V(y)}\,\mathrm{d}y \leq \big|B(x,r) \cap \Omega_M\big| + \frac{1}{1+M}\big|B(0,r)\big|.$$

One realizes then that the above sufficient criterion of discreteness coincides with the one already given in [**6**] under Assumption (3); one sees also that A.M. Molchanov's necessary condition follows from "thinness at infinity" of sublevel sets Ω_M since

$$\begin{aligned}\big|B(0,r)\big| = \big|B(x,r)\big| &= \int_{B(x,r)} \frac{\sqrt{1+V(y)}}{\sqrt{1+V(y)}}\,\mathrm{d}y \\ &\leq \Big(\int_{B(x,r)} \frac{1}{1+V(y)}\,\mathrm{d}y\Big)^{\frac{1}{2}} \Big(\int_{B(x,r)} \big(1+V(y)\big)\,\mathrm{d}y\Big)^{\frac{1}{2}}\end{aligned}$$

and then

$$\int_{B(x,r)} V(y)\,\mathrm{d}y \geqslant -\big|B(0,r)\big| + \frac{\big|B(0,r)\big|^2}{\int_{B(x,r)} \frac{1}{1+V(y)}\,\mathrm{d}y};$$

it seems that this has not been noticed in the literature on the subject.