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MÉMOIRES DE LA SMF 150

Société Mathématique de France 2017

Diffusion

Maison de la SMF	AMS
B.P. 67	P.O. Box 6248
13274 Marseille Cedex 9	Providence RI 02940
France	USA
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Tarifs 2017

Vente au numéro : 35 € ($ 52)
Abonnement électronique : 113 € ($ 170)
Abonnement avec supplément papier : 162 €, *hors Europe :* 186 € ($ 279)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Mémoires de la SMF
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
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ISSN 0249-633-X (print) 2275-3230 (electronic)

ISBN 978-2-85629-849-7

Stéphane SEURET
Directeur de la publication

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Publié avec le concours du Centre National de la Recherche Scientifique

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2010 *Mathematics Subject Classification*. — 83C05, 35L15, 83C99.

***Key words and phrases*. —** Einstein gravity, modified gravity, Cauchy problem, augmented conformal formulation.

The authors were partially supported by the Agence Nationale de la Recherche (ANR) through the grant 06-2-134423 and ANR SIMI-1-003-01. Part of this research was done when the first author (PLF) was a visiting professor for the Fall Semester 2013 at the Mathematical Sciences Research Institute (Berkeley) and was supported by the National Science Foundation under Grant No. 0932078 000.

THE MATHEMATICAL VALIDITY OF THE f(R) THEORY OF MODIFIED GRAVITY

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***Abstract.* —** We investigate the Cauchy problem for the $f(R)$ theory of modified gravity, which is a generalization of Einstein's classical theory of gravitation. The integrand of the Einstein-Hilbert functional is the scalar curvature R of the spacetime, while, in modified gravity, it is a nonlinear function $f(R)$ so that, in turn, the field equations of the modified theory involve *up to fourth-order* derivatives of the unknown spacetime metric. We introduce here a *formulation of the initial value problem in modified gravity* when initial data are prescribed on a spacelike hypersurface. We establish that, in addition to the induced metric and second fundamental form (together with the initial matter content, if any), an initial data set for modified gravity must also provide one with the *spacetime scalar curvature* and its first-order time-derivative. We propose an *augmented conformal formulation* (as we call it), in which the spacetime scalar curvature is regarded as an *independent variable.* In particular, in the so-called wave gauge, we prove that the field equations of modified gravity are equivalent to a coupled system of *nonlinear wave-Klein-Gordon equations* with defocusing potential. We establish the consistency of the proposed formulation, whose main unknowns are the conformally-transformed metric and the scalar curvature (together with the matter fields) and we establish the existence of a maximal globally hyperbolic Cauchy development associated with any initial data set with sufficient Sobolev regularity when, for definiteness, the matter is represented by a massless scalar field. We analyze the so-called *Jordan coupling* and work with the so-called *Einstein metric*, which is conformally equivalent to the physical metric — the conformal factor depending upon the unknown scalar curvature. A main result in this paper is the derivation of quantitative estimates in suitably defined functional spaces, which are uniform in term of the nonlinearity $f(R)$ and show that spacetimes of modified gravity are *'close' to Einstein spacetimes*, when the defining function $f(R)$ is 'close' to the Einstein-Hilbert integrand R. We emphasize that this is a highly singular limit problem, since the field equations under consideration are fourth-order in the metric, while the Einstein equations are second-order only. In turn, our analysis provides the first mathematically rigorous validation of the theory of modified gravity.

***Résumé* (*Validité mathématique de la théorie $f(R)$ de la gravité modifiée*)**
Nous étudions le problème de Cauchy pour la théorie $f(R)$ de la gravité modifiée, laquelle généralise la théorie classique de gravitation due à Einstein. L'intégrant de la fonctionnelle d'Einstein-Hilbert est la courbure scalaire de l'espace-temps, tandis que, dans la théorie de la gravité modifiée, l'intégrant est une fonction nonlinéaire $f(R)$, et les équations de champ sont d'ordre quatre par rapport aux dérivées de la métrique inconnue. Nous introduisons ici une formulation du problème de valeurs initiales pour la gravité modifiée, lorsque des données sont prescrites sur une hypersurface de type espace. Nous établissons que, en plus de la métrique induite et de la deuxième forme fondamentale, il est nécessaire de se donner la courbure de l'espace-temps et sa dérivée première. Nous proposons alors une « formulation conforme augmentée » dans laquelle la courbure scalaire est une inconnue indépendante supplémentaire. Dans la jauge des ondes (ou jauge harmonique), nous démontrons que les équations de champ forment un système couplé nonlinéaire d'équations d'ondes et d'équations de Klein-Gordon. Nous établissons une propriété de consistance pour ce système dont les inconnues sont la métrique conforme et la courbure scalaire, et nous démontrons l'existence d'un développement de Cauchy maximal lorsque les données initiales ont une régularité de type Sobolev et que la matière est décrite par un champ scalaire sans masse. Nous analysons le « couplage de Jordan » dans la métrique d'Einstein qui est conformément équivalente à la métrique physique. Nous obtenons des estimées de type énergie dans des espaces fonctionnels à poids ; ces estimées sont *uniformes* par rapport à la nonlinéarité $f(R)$ et nous permettent de valider rigoureusement la limite singulière $f(R) \to R$. Nous montrons ainsi que le système d'ordre quatre de la gravité modifiée converge vers le système d'ordre 2 de la gravité d'Einstein. Ce travail établit donc la validité mathématique de la théorie de la gravité modifiée.

CONTENTS

CHAPTER 1

INTRODUCTION

In recent years, new observational data have suggested that alternative theories of gravity, based on extensions of Einstein's field equations of general relativity, may be relevant in order to explain the accelerated expansion of the Universe as well as certain instabilities observed in galaxies —without explicitly introducing notions such as 'dark energy' or 'dark matter'. Among these theories, the so-called $f(R)$*-theory of modified gravity* (associated with a prescribed function $f(R)$ of the scalar curvature R) was recognized as a physically viable alternative to Einstein's theory. Despite the important role played by this theory in physics, the corresponding field equations have not been investigated by mathematicians yet. This is due to the fact that the modified gravity equations are significantly more involved than the Einstein equations: they contain up to *fourth-order derivatives* of the unknown metric, rather than solely second-order derivatives. Extensive works are available in the physical and numerical literature [**3**], [**4**], [**7**], [**8**], [**13**], [**23**], [**24**], [**25**]. The study of the well-posedness for this theory was also investigated earlier for instance in [**9**] by taking advantage of an equivalence with the Brans-Dicke theory. Furthermore, the function f is sometimes taken to be singular (and this leads to a further difficulty [**5**]), but here we assume this function to be regular.

Our purpose in this article is to initiate a rigorous mathematical study of the modified gravity equations and, specifically,

– to introduce a *notion of initial data set in modified gravity,*
– to describe an *initial value formulation* from an arbitrary spacelike hypersurface,
– to establish the *existence of a globally hyperbolic maximal development* associated with a given initial data set,
– and, importantly, to *provide a rigorous validation* that the modified gravity theory is an 'approximation' of Einstein's theory, in sense that we will make precise with quantitative estimates. For definiteness, we will deal with asymptotically flat solutions, although our arguments are purely local and could be formulated

in a domain of dependence of any initial data set. Our setting is appropriate in order to address the global nonlinear stability of Minkowski spacetime which we establish in the series of papers [**18**]–[**22**].

As already mentioned, in addition to the (second-order) Ricci curvature terms arising in the Einstein equations, the field equations of the $f(R)$-theory involve fourth-order derivatives of the metric and, more precisely, second-order derivatives of the scalar curvature. The corresponding system of partial differential equations (after a suitable choice of gauge) consists of a system of *nonlinear wave equations*, which is significantly more involved than the corresponding system derived from Einstein's equations. Yet, a remarkable mathematical structure is uncovered in the present work, which we refer to as the *augmented conformal formulation*:

– we introduce a *conformally equivalent metric* based on a conformal factor that depends upon the (unknown) scalar curvature,
– we proceed by introducing an *extended system* in which the metric and its scalar curvature are regarded as *independent unknowns*,
– we then establish the *well-posedness* of the initial value problem for this augmented formulation,
– and we finally explain how to recover the solutions to the original system of modified gravity.

Before we present our results in further details, let us first to recall that Einstein's theory is based on *Hilbert-Einstein's action*

$$\mathcal{A}_{\mathrm{HE}}[\phi, g] := \int_M \Big(\frac{R_g}{16\pi} + L[\phi, g] \Big) \, dV_g \tag{1.0.1}$$

associated with a $(3+1)$-dimensional spacetime (M, g) with Lorentzian signature $(-,+,+,+)$ whose canonical volume form is denoted by $dV = dV_g$. Here, and thereafter, we denote by $\mathrm{Rm} = \mathrm{Rm}_g$, $\mathrm{Ric} = \mathrm{Ric}_g$, and $R = R_g$ the Riemann, Ricci, and scalar curvature of the metric g, respectively. Observe that the above functional $\mathcal{A}_{\mathrm{EH}}[g]$ is determined from the scalar curvature R_g and a Lagrangian $L[\phi, g]$, the latter term describing the matter content represented by one or several fields ϕ defined on M.

It is well-known that critical metrics for the action $\mathcal{A}_{\mathrm{EH}}[g]$ (at least formally) satisfy Einstein's equation

$$G_g := \mathrm{Ric}_g - \frac{R_g}{2} g = 8\pi \, T[\phi, g], \tag{1.0.2}$$

in which the right-hand side [1]

$$T_{\alpha\beta}[\phi, g] := -2 \frac{\delta L}{\delta g^{\alpha\beta}}[\phi, g] + g_{\alpha\beta} \, L[\phi, g] \tag{1.0.3}$$

1. Greek indices $\alpha, \beta = 0, 1, 2, 3$ represent spacetime indices.