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DIOPHANTINE APPLICATIONS OF GEOMETRIC INVARIANT THEORY

MÉMOIRES DE LA SMF 152

Société Mathématique de France 2017

Diffusion

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AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs 2017

Vente au numéro : 35 € ($ 52)
Abonnement électronique : 113 € ($ 170)
Abonnement avec supplément papier : 162 €, *hors Europe :* 186 € ($ 279)
Des conditions spéciales sont accordées aux membres de la SMF.

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ISSN 0249-633X (print) 2275-3230 (electronic)

ISBN 978-2-85629-865-7

Stéphane SEURET
Directeur de la publication

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Publié avec le concours du Centre National de la Recherche Scientifique

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2010 ***Mathematics Subject Classification.*** --- Primary 14L24, 14G40, 11K60, 14G22.

Key words and phrases. --- Geometric Invariant Theory, diophantine approximation, Roth's Theorem, Berkovich spaces, height of semi-stable points.

DOI. --- 10.24033/msmf.460.

Partially supported by ANR research project "Positive" ANR-2010-BLAN-0119-01.

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Abstract. --- This text consists of two parts. In the first one we present a proof of Thue-Siegel-Roth's Theorem (and its more recent variants, such as those of Lang for number fields and that "with moving targets" of Vojta) as an application of Geometric Invariant Theory (GIT). Roth's Theorem is deduced from a general formula comparing the height of a semi-stable point and the height of its projection on the GIT quotient. In this setting, the role of the zero estimates appearing in the classical proof is played by the geometric semi-stability of the point to which we apply the formula.

In the second part we study heights on GIT quotients. We generalise Burnol's construction of the height and refine diverse lower bounds of the height of semi-stable points established to Bost, Zhang, Gasbarri and Chen. The proof of Burnol's formula is based on a non-archimedean version of Kempf-Ness theory (in the framework of Berkovich analytic spaces) which completes the former work of Burnol.

***Résumé* (*Applications diophantiennes de la théorie géométrique des invariants*)**

Ce texte est constitué de deux parties. Dans la première nous présentons une preuve du théorème de Thue-Siegel-Roth (et des variantes plus récentes, comme celle de Lang pour le corps de nombres et celle with moving targets de Vojta) basée sur la théorie géométrique des invariants (GIT). Le théorème de Roth est déduit d'une formule reliant la hauteur d'un point semi-stable et la hauteur de sa projection dans le quotient GIT. Dans ce cadre, le rôle du « lemme des zéros » présent dans la preuve classique est joué par la semi-stabilité géométrique du point auquel on applique la formule.

Dans la deuxième partie nous étudions la hauteur sur les quotients GIT. Nous généralisons la construction de Burnol de cette hauteur et nous améliorons plusieurs minorations de la hauteur de point semi-stables précédemment établies par Bost, Zhang Gasbarri et Chen. La preuve de la formule de Burnol porte sur une version non-archimédienne de la théorie de Kempf-Ness (dans le langage de la géométrie analytique de Berkovich), qui complète le travail antérieur de Burnol.

CONTENTS

INTRODUCTION

In its original form, Roth's Theorem states that given a real algebraic number $\alpha \in \mathbb{R}$ which is not rational and a real number $\kappa > 2$, there exist only finitely many rational numbers $p/q \in \mathbb{Q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{|q|^{\kappa}}$$

where p, q are coprime integers.

The general strategy to prove Roth's Theorem stems back to the work of Thue. The main ingredient is the construction of an "auxiliary" polynomial f in several variables which vanishes at high order at $(\alpha, \ldots, \alpha)$: the crucial step is to prove that it does not vanish too much at rational points which "approximate" $(\alpha, \ldots, \alpha)$.

The original argument of Roth (generalizing those of Thue, Siegel and Gel'fond) involves arithmetic considerations about the height of the rational approximations. On the other hand, in the work of Dyson — who proved an earlier version of Roth's Theorem — the non-vanishing result (usually called "Dyson's Lemma") takes place over the complex numbers: being free from arithmetic constraints, it is said to be of geometric nature. The task to generalize Dyson's Lemma from 2 to several variables was accomplished by Esnault-Viehweg [**28**]; afterwards Nakamaye [**50**] was able to give a proof of it relying on a variant of Faltings' Product Theorem and "elementary" concepts of intersection theory.

The advantage of having a geometric proof of Dyson's Lemma was exploited by Bombieri in the remarkable paper [**7**]: he showed that these methods lead to new effective results in diophantine approximation available before only through the linear forms of logarithms of Baker.

Using an arithmetic variant of the Product Theorem, Faltings and Wüstholz [**30**] gave a new proof of Schmidt's Subspace Theorem, sensibly different from the original one. Their Zero Lemma, as in Roth and Schmidt, is of arithmetic nature. Their proof involves a notion of semi-stability for filtered vector spaces (see also [**29**]). The role played by semi-stability is anyway rather different from the one in the present paper: here it collects all the geometric informations coming from Dyson's Lemma (hence from the Product Theorem); in their paper it represents a combinatorial assumption that permits to perform an inductive step based on the Product Theorem.

Inspired by work of Osgood [**53**] and Steinmetz [**65**] Vojta proved in [**69**] a generalised version of Roth's Theorem — called "with moving targets" — where the algebraic point can vary along with the rational approximations. Its proof is based on the use of Schmidt's Subspace Theorem. However it has been noticed by Bombieri and Gubler [**8**, Theorem 6.5.2 and §6.6] that the techniques employed to prove Roth's Theorem suffice to prove the version "with moving targets" without recurring to Schmidt's Subspace Theorem.

The study of the interplay between Geometric Invariant Theory and height functions (in the context of Arakelov geometry) has started more than twenty years ago with the work of several authors.

Burnol [**20**] defined a height function on the GIT quotient of a projective space by a reductive group and he expressed it in terms as the sum of the height on the projective space and of local error terms.

Bost [**12**], [**13**], Zhang [**75**] and Soulé [**64**] proved several lower bounds on the height of semi-stable points (in some explicit representations) and used them to give lower bounds on the height of semi-stable varieties (*e.g.* semi-stable curves, abelian varieties...).

Gasbarri [**32**] was able to free the arguments of Bost and Zhang from the constraint of knowing explicitly the representation of GL_n. Chen [**22**] proved an explicit variant of this type of lower bounds and used it — inspired by work of Ramanan-Ramanathan [**56**] and Totaro [**67**] — to study the semi-stability of the tensor product of hermitian vector bundles over a ring of integers.

In the first chapter of this text, we show how a simple version of this general lower bound on the height of (geometrically) semi-stable point leads to a general lower bound on the height of suitable families of points $(x_1, \ldots, x_n)$ and $(a_1, \ldots, a_n)$ in $\mathbb{P}^1(K)^n$ and $\mathbb{P}^1(K')^n$ respectively to the diverse v-adic distances (where K is a number field and K' is an extension of degree ≥ 2). This lower bound, which constitutes the main result of the present note, has been established in the case $n = 2$ by Bombieri [**7**, Theorem 2], is effective and implies the version of Roth's Theorem we present here.

Let us discuss briefly the content of the chapters. For the precise statement of the results we refer the reader to the first section of each chapter.

In Chapter 1 we introduce the basic tools of Geometric Invariant Theory that are needed in order to deduce Roth's Theorem. The results expounded in this chapter will be refined in Chapter 4, but we preferred to give a succinct and self-containted account for the reader interested to the proof of Roth's Theorem in Chapter 2. It may also serve the reader interested in Chapters 3 and 4 as an introduction to the results to be improved.

In Chapter 2 we prove Roth's Theorem (along with some more recent variants) as a consequence of the Fundamental Formula in Chapter 2 (applied to a suitably chosen "moduli problem").

In Chapter 3 we investigate a variant of the results of Kempf-Ness [**46**] for complex and non-archimedean geometry.

In Chapter 4 we study deeply the height on the quotient and prove (some of) the desired refinements of the results of Chapter 1.

An effort has been made in order to keep the different chapters independent one from the other. In Chapter 2 the only references to Chapter 1 are in section 2.3, while in Chapter 4 the needed facts from Chapter 3 are recalled in section 1. We invite to read the chapters separately.

Acknowledgements

The material expounded in this article is part of my doctoral thesis supervised J.-B. Bost. It is a pleasure for me to thank him for his guidance and his steady encouragement. During the preparation of the present article I have been stimulated by discussions with several people: I warmly thank A. Chambert-Loir, A. Ducros, C. Gasbarri, M. Nakamaye and A. Thuillier. Finally, I want to sincerely thank the referees for their careful reading and the series of remarks that sensibly improved the quality of the present paper.