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FACTORIZATION OF NON-SYMMETRIC OPERATORS AND EXPONENTIAL H-THEOREM

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FACTORIZATION OF NON-SYMMETRIC OPERATORS AND EXPONENTIAL H-THEOREM

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Abstract. — We present an abstract method for deriving decay estimates on the resolvents and semigroups of non-symmetric operators in Banach spaces in terms of estimates in another smaller reference Banach space. This applies to a class of operators writing as $\mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ is bounded, $\mathcal{B}$ is dissipative and the two parts satisfy a semigroup commutator condition of regularization. The core of the method is a high-order quantitative factorization argument on the resolvents and semigroups. We then apply this approach to the Fokker-Planck equation, to the kinetic Fokker-Planck equation in the torus, and to the linearized Boltzmann equation in the torus.

We finally use this information on the linearized Boltzmann semigroup to study perturbative solutions for the nonlinear Boltzmann equation. We introduce a non-symmetric energy method to prove nonlinear stability in this context in $L^1_v L^\infty_x(1 + |v|^k)$, $k > 2$, with sharp rate of decay in time. Our result drastically improves the class of functions considered in the literature, it also provides optimal rate of convergence and our proof is constructive.

As a consequence of these results, we obtain the first constructive proof of exponential decay, with sharp rate, towards global equilibrium for the full nonlinear Boltzmann equation for hard spheres, conditionally to some smoothness and (polynomial) moment estimates. This improves the result in [46] where polynomial rates at any order were obtained, and solves the conjecture raised in [119], [43], [110] about the optimal decay rate of the relative entropy in the H-theorem.

Résumé. — Nous présentons une méthode abstraite pour démontrer des estimations de décroissance sur les résolvantes et les semi-groupes d'opérateurs non-symétriques dans des espaces de Banach, à partir d'estimations dans un autre espace de Banach de référence plus petit. Cette méthode s'applique à une classe d'opérateurs s'écrivant $\mathcal{A} + \mathcal{B}$ avec $\mathcal{A}$ borné et $\mathcal{B}$ dissipatif, et sous une condition de régularisation sur un commutateur au niveau des semi-groupes. Le cœur

de la méthode est un argument de factorisation quantifiée d'ordre élevé sur les résolvantes et semi-groupes. Nous appliquons ensuite cette approche à l'équation de Fokker-Planck, à l'équation de Fokker-Planck cinétique dans le tore, ainsi qu'à l'équation de Boltzmann linéarisée dans le tore. Nous exploitons enfin l'information ainsi obtenue sur le semi-groupe linéarisé de Boltzmann pour étudier les solutions perturbatives du problème non-linéaire. Nous introduisons une méthode d'énergie non-symétrique pour prouver la stabilité non-linéaire dans ce contexte dans $L^1_v L^\infty_x(1+|v|^k)$, $k > 2$, avec taux de décroissance en temps précis. Notre résultat améliore grandement les résultats précédents par la classe de fonctions considérée, il fournit également le taux de convergence optimal, et la preuve est constructive. Comme conséquence de ces résultats, nous obtenons la première preuve constructive de la relaxation exponentielle vers l'équilibre avec taux optimal pour l'équation de Boltzmann non-linéaire complète, pour des interactions de type sphères dures, conditionnellement à des bornes de régularité et de moments polynômiaux. En particulier, cela étend les résultats de [**46**], où des taux de relaxation polynômiaux avaient démontrés à tout ordre, et résoud la conjecture formulée dans [**119**], [**43**], [**110**] sur le taux de relaxation optimal de l'entropie relative dans le théorème H.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. The problem at hand

This book deals with

(i) the study of resolvent estimates and decay properties for a class of generators and associated semigroups in general Banach spaces, and

(ii) the study of relaxation to equilibrium for some kinetic evolution equations, which makes use of the previous abstract tools.

Let us give a brief sketch of the first problem. Consider two Banach spaces $E \subset \mathcal{E}$, and two C_0-semigroup generators L and $\mathcal{L}$ respectively on E and $\mathcal{E}$ with spectrum $\Sigma(L), \Sigma(\mathcal{L}) \subsetneq \mathbb{C}$. Denote $S(t)$ and $\mathcal{S}(t)$ the two associated semigroups respectively in E and $\mathcal{E}$. Further assume that $\mathcal{L}_{|E} = L$, and E is dense in $\mathcal{E}$. The theoretical question we address in this work is the following:

> *Can one derive quantitative informations on $\Sigma(\mathcal{L})$ and $\mathcal{S}(t)$ in terms of informations on $\Sigma(L)$ and $S(t)$?*

We provide here an answer for a class of operators $\mathcal{L}$ which split as

$$\mathcal{L} = \mathcal{A} + \mathcal{B},$$

where the spectrum of $\mathcal{B}$ is well localized and the iterated convolution $(\mathcal{A}\mathcal{S}_{\mathcal{B}})^{*n}$ maps $\mathcal{E}$ to E with proper time-decay control for some $n \in \mathbb{N}^*$. We then prove that

(i) $\mathcal{L}$ inherits most of the spectral gap properties of L;

(ii) explicit estimates on the rate of decay of the semigroup $\mathcal{S}(t)$ can be computed from the ones on $S(t)$.

The core of the proposed method is a robust factorization argument on the resolvents and semigroups, reminiscent of the Dyson series.

In a second part of this book, we then show that the kinetic Fokker-Planck operator and the linearized Boltzmann operator for hard sphere interactions satisfy the above abstract assumptions, and we thus extend the known spectral-gap properties from the standard linearization space (an L^2 space with Gaussian weight prescribed by the equilibrium) to larger Banach spaces (for example L^p with polynomial decay). It is worth mentioning that the proposed method provides optimal rate of decay and there is no loss of accuracy in the extension process from E to $\mathcal{E}$ (as would be the case in, say, interpolation approaches).

Proving the abstract assumption requires significant technical efforts for the Boltzmann equation and leads to the introduction of new tools: some specific estimates on the collision operator, some iterated averaging lemma and a nonlinear non-symmetric energy method. All together, we are able to prove nonlinear stability of Gaussian equilibrium and of space homogeneous solutions for the Boltzmann equation for hard spheres interactions in the torus in a $L^1_v L^\infty_x(1 + |v|^k)$, $k > 2$, framework with sharp rate of decay in time. That result drastically improves the class of functions considered in the literature since the seminal work by Ukai [**120**] and provides (for the very first time) optimal rate of decay. The method of proof is also completely constructive.

1.2. Motivation

The motivation for the abstract part of this book, *i.e.* enlarging the functional space where spectral properties are known to hold for a linear operator, comes from nonlinear PDE analysis.

The first motivation is when the linearized stability theory of a nonlinear PDE is *not* compatible with the nonlinear theory. More precisely, the natural function space where the linearized equation is well-posed and stable, with nice symmetric or skew-symmetric properties for instance, is "too small" for the nonlinear PDE in the sense that no well-posedness theorem is known (and conjectured to be false) in such a space. This is the case for the classical Boltzmann equation and therefore it is a key obstacle in obtaining perturbative result in natural physical spaces and connecting the nonlinear results to the perturbative theory.

This is related to the famous H-theorem of Boltzmann. The natural question of understanding mathematically the H-theorem was emphasized by Truesdell and Muncaster [**119**, pp. 560–561] thirty years ago:

> "*Much effort has been spent toward proof that place-dependent solutions exist for all time. [...] The main problem is really to discover and specify the circumstances that give rise to solutions which persist forever. Only after having done that can we expect to construct proofs that such solutions exist, are unique, and are regular.*"