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# BRACKETS IN THE PONTRYAGIN ALGEBRAS OF MANIFOLDS

**Gwénaël Massuyeau, Vladimir Turaev**

***Abstract.*** — Given a smooth oriented manifold $M$ with non-empty boundary, we study the Pontryagin algebra $A = H_*(\Omega)$ where $\Omega$ is the space of loops in $M$ based at a distinguished point of $\partial M$. Using the ideas of string topology of Chas-Sullivan, we define a linear map

$$\{\{-,-\}\} : A \otimes A \longrightarrow A \otimes A$$

which is a double bracket in the sense of Van den Bergh satisfying a version of the Jacobi identity. For $\dim(M) \geq 3$, the double bracket $\{\{-,-\}\}$ induces Gerstenhaber brackets in the representation algebras associated with $A$. This extends our previous work on the case $\dim(M) = 2$ where $A = H_0(\Omega)$ is the group algebra of the fundamental group $\pi_1(M)$ and the double bracket $\{\{-,-\}\}$ induces the standard Poisson brackets on the moduli spaces of representations of $\pi_1(M)$.

***Résumé*** **(Crochets dans les algèbres de Pontryagin des variétés)**

Étant donnée une variété $M$, lisse, orientée et à bord non-vide, nous étudions l'algèbre de Pontryagin $A = H_*(\Omega)$ où $\Omega$ désigne l'espace des lacets dans $M$ basés en un point distingué de $\partial M$. En utilisant les idées de la topologie des cordes de Chas et Sullivan, nous définissons une application linéaire

$$\{\{-,-\}\} : A \otimes A \longrightarrow A \otimes A$$

qui est un crochet double au sens de Van den Bergh et satisfait une version de l'identité de Jacobi. Lorsque $\dim(M) \geq 3$, le crochet double $\{\{-,-\}\}$ induit des crochets de Gerstenhaber sur les algèbres de représentations associées à $A$. Ceci étend notre précédent travail sur le cas $\dim(M) = 2$ où $A = H_0(\Omega)$ est l'algèbre de groupe du groupe fondamental $\pi_1(M)$ et le crochet double $\{\{-,-\}\}$ induit les crochets de Poisson habituels sur les espaces de modules de représentations de $\pi_1(M)$.

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# INTRODUCTION

A remarkable feature of an oriented surface $\Sigma$ discovered by Goldman [**20**], [**21**] is a natural Lie bracket in the vector space generated by the free homotopy classes of loops in $\Sigma$. If $\Sigma$ is connected and closed, then Goldman's Lie bracket arises from a symplectic structure on the moduli space of representations of the fundamental group $\pi = \pi_1(\Sigma)$ in a Lie group $G$. This space $\mathrm{Hom}(\pi, G)/G$ consists of the conjugacy classes of homomorphisms $\pi \to G$. The resulting symplectic structure incorporates the classical Kähler forms on the Teichmüller space ($G = \mathrm{PSL}(2, \mathbb{R})$), on the Jacobi variety ($G = \mathrm{U}(1)$), and on the Narasimhan–Seshadri moduli spaces of semistable vector bundles ($G = \mathrm{U}(N)$ with $N \geq 1$). Goldman's construction also yields the Atiyah-Bott symplectic structure determined by a compact Lie group and a non-degenerate $\mathrm{ad}$-invariant symmetric bilinear form on its Lie algebra. If $\Sigma$ is connected and $\partial\Sigma \neq \emptyset$, then similar methods yield a weaker structure, namely, a Poisson bracket in the algebra of conjugation-invariant smooth functions on $\mathrm{Hom}(\pi, G)$, see [**17**], [**23**]. This bracket extends to a quasi-Poisson bracket in the algebra of all smooth functions on $\mathrm{Hom}(\pi, G)$, see [**2**]. Analogous results hold for the general linear group $G = \mathrm{GL}_N$ over any commutative ring provided $\mathrm{Hom}(\pi, \mathrm{GL}_N)$ is treated as an affine algebraic set and smooth functions are traded for regular functions, see [**36**].

Goldman's Lie bracket for surfaces was generalized by Chas and Sullivan [**9**], [**10**] to manifolds of arbitrary dimensions. Chas and Sullivan call this area of study the "string topology". The present memoir exhibits new phenomena in string topology. We consider the Pontryagin algebras of manifolds with boundary and construct a bracket in the associated representation algebras. For surfaces, our bracket is the quasi-Poisson bracket on $\mathrm{Hom}(\pi, \mathrm{GL}_N)$ mentioned above. In dimension $\geq 3$, the representation algebras are graded, and our bracket is a Gerstenhaber bracket, *i.e.*, it satisfies the axioms of a Poisson bracket with appropriate signs. In the rest of the Introduction we focus on manifolds of dimension $\geq 3$.

We recall the concept of a representation algebra following [**39**], [**34**], [**13**]. Fix an integer $N \geq 1$ and a field $\mathbb{F}$ which will be the ground field of the algebras. Given an algebra $A$ and a commutative algebra $B$, consider the set $S = S(A, N, B)$ of all algebra homomorphisms from $A$ to the algebra $\mathrm{Mat}_N(B)$ of $(N \times N)$-matrices over $B$.

Each $a \in A$ and each pair of indices $i, j \in \{1, \ldots, N\}$ determine a mapping $a_{ij} : S \to B$ which evaluates a homomorphism $A \to \mathrm{Mat}_N(B)$ at $a$ and takes the $(i, j)$-th entry of the resulting matrix. These mappings are the "coordinates" on $S$, generating an algebra of "polynomial" $B$-valued functions on $S$. These coordinates satisfy various polynomial relations some of which are universal, *i.e.*, hold for all $B$. By definition, the $N$-th *representation algebra* $A_N$ of $A$ is generated by the symbols $\{a_{ij} \,|\, a \in A, 1 \le i, j \le N\}$ subject to those universal relations. One of the universal relations says that the generators commute, so that $A_N$ is a commutative algebra. For every commutative algebra $B$, the algebra $A_N$ projects onto the algebra of polynomial $B$-valued functions on $S(A, N, B)$ described above. We view $A_N$ as a universal form of these polynomial algebras. If $A$ is graded, then so is $A_N$.

Our construction of brackets in the representation algebras $\{A_N\}_{N \ge 1}$ is based on the technique of Van den Bergh [**42**]. He showed how to construct such brackets from a linear map

$$\{\!\{-,-\}\!\} : A \otimes A \longrightarrow A \otimes A$$

satisfying certain conditions. Van den Bergh calls such maps *double Poisson brackets*. We use the term *bibracket* for the version of double brackets used here. Also, we work in the graded setting and rather consider *Gerstenhaber bibrackets* satisfying a graded version of the Jacobi identity. We show that a Gerstenhaber bibracket $\{\!\{-,-\}\!\}$ in a graded algebra $A$ induces a Gerstenhaber bracket $\{-,-\}$ in $A_N$ for all $N \ge 1$. In terms of the generators, the bracket $\{-,-\}$ is defined as follows: for any $a, b \in A$, $i, j, u, v \in \{1, \ldots, N\}$, and any finite expansion

$$\{\!\{a, b\}\!\} = \sum_\alpha x_\alpha \otimes y_\alpha \in A \otimes A,$$

we set

$$\{a_{ij}, b_{uv}\} = \sum_\alpha (x_\alpha)_{uj} (y_\alpha)_{iv}.$$

The bracket $\{-,-\}$ is invariant under the natural actions of the group $\mathrm{GL}_N(\mathbb{F})$ and the Lie algebra $\mathrm{Mat}_N(\mathbb{F})$ on $A_N$.

Consider now a smooth oriented manifold $M$ of dimension $\ge 3$ with base point

$$\star \in \partial M \neq \emptyset.$$

Let $\Omega = \Omega_\star$ be the space of loops in $M$ based at $\star$. The graded vector space

$$A = H_*(\Omega; \mathbb{F})$$

carries an associative multiplication induced by concatenation of loops. This turns $A$ into a graded algebra, the *Pontryagin algebra* of $M$. We define a so-called *intersection bibracket* in $A$ as follows. Pick an embedded path $\varsigma : I = [0,1] \hookrightarrow \partial M$ connecting the point $\star$ to another point $\star'$. Consider any singular cycles $\kappa : K \to \Omega = \Omega_\star$ and $\lambda : L \to \Omega' = \Omega_{\star'}$. Let $D$ be the set of all tuples $(k \in K, s \in I, l \in L, t \in I)$ such that $\kappa(k)(s) = \lambda(l)(t)$. Each tuple $(k, s, l, t) \in D$ determines two loops in $M$ based at $\star$. The first loop goes along $\varsigma$ from $\star$ to $\star'$, then along the path $\lambda(l)$ from $\star' = \lambda(l)(0)$ to $\lambda(l)(t) = \kappa(k)(s)$ and then along the path $\kappa(k)$ back to $\kappa(k)(1) = \star$. The second loop goes along the path $\kappa(k)$ from $\star = \kappa(k)(0)$ to $\kappa(k)(s) = \lambda(l)(t)$, then along $\lambda(l)$