Claude Sabbah

with the collaboration of Jeng-Daw Yu

IRREGULAR HODGE THEORY

MÉMOIRES DE LA SMF 156

Société Mathématique de France 2018

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs 2018

Vente au numéro : 40 € ($ 60)
Abonnement électronique : 113 € ($ 170)
Abonnement avec supplément papier : 167 €, *hors Europe :* 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Mémoires de la SMF
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
memoires@smf.emath.fr • http://smf.emath.fr/

ISSN 0249-633X (print) 2275-3230 (electronic)

ISBN 978-2-85629-877-9

Stéphane SEURET
Directeur de la publication

MÉMOIRES DE LA SMF 156

IRREGULAR HODGE THEORY

Claude Sabbah

with the collaboration of Jeng-Daw Yu

Société Mathématique de France 2018
Publié avec le concours du Centre National de la Recherche Scientifique

C. Sabbah
CMLS, École polytechnique, CNRS, Université Paris-Saclay,
F–91128 Palaiseau cedex, France.
E-mail : Claude.Sabbah@polytechnique.edu
Url : http://www.math.polytechnique.fr/perso/sabbah

2010 *Mathematics Subject Classification*. — 14F40, 32S35, 32S40.

***Key words and phrases*. —** Holonomic D-module, mixed Hodge module, mixed twistor D-module, irregular Hodge filtration, exponential-Hodge origin, Thom-Sebastiani formula.

***DOI*. —** 10.24033/msmf.464

This research was supported by the grant ANR-13-IS01-0001-01 of the Agence nationale de la recherche. The collaboration with Jeng-Daw Yu was partly supported by the MoST/CNRS grants 106-2911-I-002-539/TWN PRC 1632.

IRREGULAR HODGE THEORY

Claude Sabbah

with the collaboration of Jeng-Daw Yu

Abstract. — We introduce the category of *irregular mixed Hodge modules* consisting of possibly irregular holonomic D-modules which can be endowed in a canonical way with a filtration, called the *irregular Hodge filtration*. Mixed Hodge modules with their Hodge filtration naturally belong to this category, as well as their twist by the exponential of any meromorphic function. This category is stable by various standard functors, which produce many more filtered objects. The irregular Hodge filtration satisfies the E_1-degeneration property with respect to any projective morphism. This generalizes some results previously obtained by H. Esnault, J.-D. Yu and the author. We also show that, modulo a condition on eigenvalues of monodromies, any rigid irreducible holonomic D-module on the complex projective line underlies an irregular pure Hodge module. In a chapter written jointly with Jeng-Daw Yu, we make explicit the case of irregular mixed Hodge structures, for which we prove in particular a Thom-Sebastiani formula.

***Résumé* (Théorie de Hodge irrégulière).** — Nous introduisons la catégorie des *modules de Hodge mixtes irréguliers* formée de D-modules holonomes à singularités éventuellement irrégulières qui peuvent être munis de manière canonique d'une filtration, dite *filtration Hodge irrégulière*. Les modules de Hodge mixtes avec leur filtration de Hodge sont naturellement des objets dans cette catégorie, de même que leur produit tensoriel avec l'exponentielle de toute fonction méromorphe. Cette catégorie est stable par plusieurs foncteurs standard, ce qui permet d'obtenir de nombreux exemples. La filtration de Hodge irrégulière satisfait à une propriété de dégénérescence en E_1 par rapport à un morphisme projectif. Ceci généralise des résultats précédemment obtenus par H. Esnault, J.-D. Yu et l'auteur. Nous montrons aussi que, modulo une condition sur les valeurs propres des monodromies, les D-modules holonomes irréductibles rigides sur la droite projective complexe sous-tendent des modules de Hodge purs irréguliers. Dans un chapitre écrit en collaboration avec Jeng-Daw Yu, nous considérons le cas des structures de Hodge mixtes irrégulières, pour lequel nous montrons en particulier une formule de Thom-Sebastiani.

CONTENTS

INTRODUCTION

Let X be a complex manifold. The category $\mathsf{MTM}(X)$ of mixed twistor $\mathscr{D}$-modules on X, introduced by T. Mochizuki [**Moc15**], represents a vast generalization of that of mixed Hodge modules on X, introduced by M. Saito [**Sai90**] and denoted by $\mathsf{MHM}(X)$. [(1)] An intermediate step to compare these categories is the category $\mathsf{MTM}^{\mathrm{int}}(X)$ of integrable mixed twistor $\mathscr{D}$-modules on X (see below). There are natural functors

$$\mathsf{MHM}(X) \longmapsto \mathsf{MTM}^{\mathrm{int}}(X) \longmapsto \mathsf{MTM}(X),$$

where the second functor forgets the integrable structure, while the first one is a natural fully faithful functor compatible with the standard functors on each category (see [**Moc15**, §13.5]). On the other hand, the category $\mathsf{MTM}^{\mathrm{int}}(X)$ contains much more objects, since the holonomic $\mathscr{D}$-module underlying a mixed Hodge module has regular singularities, while that underlying an object of $\mathsf{MTM}^{\mathrm{int}}(X)$ can have arbitrary complicated irregular singularities.

One drawback of the category $\mathsf{MTM}^{\mathrm{int}}(X)$ (or $\mathsf{MTM}(X)$) is that objects do not come with many numerical invariants, like Hodge numbers. Our aim is to introduce an intermediate category between $\mathsf{MHM}(X)$ and $\mathsf{MTM}^{\mathrm{int}}(X)$, which is not as large as $\mathsf{MTM}^{\mathrm{int}}(X)$, but so that the $\mathscr{D}$-module associated with each object is equipped with a good filtration, called the *irregular Hodge filtration*. Many interesting irregular holonomic $\mathscr{D}$-modules underlie objects in this category, in particular those generated by $\exp\varphi$, where φ is any meromorphic function on X. Some of the standard functors (pushforward by a projective morphism, pullback by a smooth morphism) extend to this category, that we call the category of (possibly) *irregular mixed Hodge modules* and that we denote by $\mathsf{IrrMHM}(X)$, and the filtered holonomic $\mathscr{D}$-module associated with any object has a good behaviour with respect to these functors. The original idea of finding such a category is due to Deligne in 1984 [**Del07b**], and has waited the development of the theory of wild twistor $\mathscr{D}$-modules [**Moc11a, Sab09**] to start a new life in [**Sab10**], before being considered from various points of view in [**Yu14, ESY17, SY15, KKP17, Moc17**].

1. Throughout this article, we consider complex mixed Hodge module and $\mathsf{MHM}(X)$ stands for $\mathsf{MHM}(X,\mathbb{C})$, see Section 1.7.b.

Let us emphasize some discrepancies between the usual Hodge filtration and the irregular Hodge filtration.

(1) A mixed Hodge module can naturally be seen as an object of $\mathsf{IrrMHM}(X)$, and in such a way its Hodge filtration is the irregular Hodge filtration. However, the properties of the irregular Hodge filtration of the $\mathscr{D}$-module underlying an object of $\mathsf{IrrMHM}(X)$ are not enough to characterize this object as such, i.e., are not constitutive of the definition of an object of $\mathsf{IrrMHM}(X)$, while the Hodge filtration is one of the fundamental objects used to define a Hodge module. In other words, the irregular Hodge filtration is only a byproduct of the definition of the category $\mathsf{IrrMHM}(X)$, and the main properties of this category are obtained from those of $\mathsf{MTM}^{\mathrm{int}}(X)$.

(2) Recall that the behaviour of the Hodge filtration with respect to nearby cycles is one of the fundamental properties used to define the category $\mathsf{MHM}(X)$. In contrast, we do not exhibit any such property for the irregular Hodge filtration. It is reasonable to expect that a similar behaviour occurs with respect to a divisor along which the mixed twistor $\mathscr{D}$-module is *tame*, but we do not even have a conjectural statement in general.

The *rescaling operation* will be used to define the category $\mathsf{IrrMHM}(X)$. It has been defined in the framework of TERP structures in [**HS07**]. This rescaling operation is the main ingredient in order to define the category ${}_{\iota}\mathsf{MTM}^{\mathrm{resc}}(X)$. However, before doing so, we need to modify the presentation of objects in $\mathsf{MTM}^{\mathrm{int}}(X)$: while it is easy to define the rescaling of an integrable $\mathscr{R}_{\mathscr{X}}$-module (with $\mathscr{X} := X \times \mathbb{C}_z$) by adding a new parameter $\theta \in \mathbb{C}^*$ and changing z to z/θ, the pairing is not rescalable, since it is defined only on $X \times \boldsymbol{S}$ ($\boldsymbol{S} = \{|z| = 1\}$), and the rescaling operation does not preserve $\boldsymbol{S}$. We are therefore led to give another description of an object in $\mathsf{MTM}^{\mathrm{int}}(X)$, where the pairing ${}_{\iota}C$ is now ι-sesquilinear ($\iota : z \mapsto -z$) and is defined on $\mathscr{X}^{\circ} := X \times \mathbb{C}_z^*$ and not only on $X \times \boldsymbol{S}$. The purpose of Chapter 1, which is essential for simply giving the definition of the rescaling operation, is to define this category ${}_{\iota}\mathsf{MTM}^{\mathrm{int}}(X)$ and to show the equivalence ${}_{\iota}\mathsf{MTM}^{\mathrm{int}}(X) \simeq \mathsf{MTM}^{\mathrm{int}}(X)$. (Note that the integrability property is important and we cannot argue without it.)

***Remark 0.1* (on the notation). —** We usually denote an object of MTM as $(\mathscr{T}, W_{\bullet})$, but we sometimes shorten the notation as $\mathscr{T}$, when the context is clear.

***The main theorems*. —** The category ${}_{\iota}\mathsf{MTM}^{\mathrm{int}}(X)$ comes equipped with standard functors which are compatible with those defined for $\mathsf{MTM}^{\mathrm{int}}(X)$ in [**Moc15**], as shown in Chapter 1:

- The duality functor $\boldsymbol{D}_X$.
- For a projective morphism $f : X \to Y$ between smooth complex manifolds, the pushforward functors $f_{\dagger}^{k} = \mathscr{H}^k f_{\dagger}$.
- For a smooth morphism $g : X \to Y$, the pullback functor g^+.
- The external product ${}^{z}\boxtimes$ (a bi-functor).