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A COMMUTATIVE $\mathbb{P}^1$-SPECTRUM REPRESENTING MOTIVIC COHOMOLOGY OVER DEDEKIND DOMAINS

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A COMMUTATIVE $\mathbb{P}^1$-SPECTRUM REPRESENTING MOTIVIC COHOMOLOGY OVER DEDEKIND DOMAINS

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Abstract. – We construct a motivic Eilenberg-MacLane spectrum with a highly structured multiplication over general base schemes which represents Levine's motivic cohomology, defined via Bloch's cycle complexes, over smooth schemes over Dedekind domains. Our method is by gluing p-completed and rational parts along an arithmetic square. Hereby the finite coefficient spectra are obtained by truncated étale sheaves (relying on the now proven Bloch-Kato conjecture) and a variant of Geisser's version of syntomic cohomology, and the rational spectra are the ones which represent Beilinson motivic cohomology.

As an application the arithmetic motivic cohomology groups can be realized as Ext-groups in a triangulated category of motives with integral coefficients.

Our spectrum is compatible with base change giving rise to a formalism of six functors for triangulated categories of motivic sheaves over general base schemes including the localization triangle.

Further applications are a generalization of the Hopkins-Morel isomorphism and a structure result for the dual motivic Steenrod algebra in the case where the coefficient characteristic is invertible on the base scheme.

CONTENTS

CHAPTER 1

INTRODUCTION

This paper furnishes the construction of a motivic Eilenberg-MacLane spectrum in mixed characteristic. One of our main purposes is to use this spectrum for the definition of triangulated categories of motivic sheaves with integral (and thus also arbitrary) coefficients over general base schemes. These categories will satisfy properties combining and expanding on properties of triangulated categories of motives which have already been constructed. In [**7**] Cisinski-Déglise develop a theory of Beilinson motives yielding a satisfying theory of motives with rational coefficients over general base schemes. This theory is equivalent to an approach due to Morel where one considers modules over the positive rational sphere spectrum, see loc. cit. Voevodsky constructed triangulated categories of motives over a (perfect) field ([**54**], [**39**]) in which integral motivic cohomology of smooth schemes is represented. In the Cisinski-Déglise/Morel category over a regular base the resulting motivic cohomology are Adams-graded pieces of rationalized K-theory, which fits with the envisioned theory of Beilinson. In [**44**] modules over the motivic Eilenberg-MacLane spectrum over a field are considered and it is proved that those are equivalent to Voevodsky's triangulated categories of motives in the characteristic 0 case. This result has recently been generalized to perfect fields [**28**] where one has to invert the characteristic of the base field in the coefficients. Étale motives are developed in [**2**] and [**6**].

We build upon these works and construct motivic categories using motivic stable homotopy theory. More precisely we define objects with a (coherent) multiplication in the category of $\mathbb{P}^1$-spectra over base schemes and consider as in [**44**] their module categories. The resulting homotopy categories are defined to be the categories of motivic sheaves.

This family of commutative ring spectra is cartesian, i.e., for any map between base schemes $X \to Y$ the pullback of the ring spectrum over Y compares via an equivalence to the ring spectrum over X. This is equivalent to saying that all spectra pull back from $\mathrm{Spec}(\mathbb{Z})$. We thus give an affirmative answer to a version of a conjecture due to Voevodsky [**56**, Conjecture 17].

To ensure good behavior of our construction our spectra have to satisfy a list of desired properties. Over fields the spectra coincide with the usual motivic Eilenberg-MacLane spectra (this ensures that over fields usual motivic cohomology is represented in our categories of motivic sheaves). Rationally we recover the theory of Beilinson motives, because the rationalizations of our spectra are isomorphic to the respective Beilinson spectra, and there is a relationship to Levine's motivic cohomology defined using Bloch's cycle complexes in mixed characteristic ([**36**]).

To ensure all of that we first construct a spectrum over any Dedekind domain $\mathscr{D}$ of mixed characteristic satisfying the following properties: It represents Bloch-Levine's motivic cohomology of smooth schemes over $\mathscr{D}$ (Corollary 7.19), it pulls back to the usual motivic Eilenberg-MacLane spectrum with respect to maps from spectra of fields to the spectrum of $\mathscr{D}$ (Theorem 8.22) and it is an E_∞-ring spectrum. (We remark that such an E_∞-structure can always be strictified to a strict commutative monoid in symmetric $\mathbb{P}^1$-spectra by results of [**24**].)

The latter property makes it possible to consider the category of highly structured modules over pullbacks of the spectrum from the terminal scheme (the spectrum of the integers), thus defining triangulated categories of motivic sheaves $\mathrm{DM}(X)$ over general base schemes X such that over smooth schemes over Dedekind domains of mixed characteristic the Ext-groups compute Bloch-Levine's motivic cohomology (Corollary 7.20, using Theorem 8.25). For general base schemes we define motivic cohomology to be represented by our spectrum, i.e.,

$$H^i_{\mathrm{mot}}(X,\mathbb{Z}(n)) := \mathrm{Hom}_{\mathsf{SH}(X)}(\mathbf{1},\Sigma^{i,n} f^*\mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}) \cong \mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Z}(0),\mathbb{Z}(n)[i]).$$

Here $f\colon X \to \mathrm{Spec}(\mathbb{Z})$ is the structure morphism, $\mathsf{M}\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ is our spectrum over the integers and $\mathbf{1}$ is the sphere spectrum (the unit with respect to the smash product) in the stable motivic homotopy category $\mathsf{SH}(X)$. By the base change property these cohomology groups coincide with Voevodsky's motivic cohomology if X is smooth over a field. We note that the ring structure on our Eilenberg-MacLane spectrum gives the (bigraded) motivic cohomology groups a (graded commutative) ring structure, a property which was (to the knowledge of the author) missing for Levine's motivic cohomology. In particular we obtain a product structure on Chow groups of smooth schemes over Dedekind domains.

By the work of Ayoub [**1**] the base change property enables one to get a full six functor formalism for these categories of motivic sheaves including the localization triangle (Theorem 9.1).

We remark that the spectrum we obtain gives rise to motivic complexes over any base scheme X. More precisely one can extract objects $\mathbb{Z}(n)^X$ in the derived category of Zariski sheaves on the category of smooth schemes over X representing our motivic cohomology. There are unital, associative and commutative multiplication maps $\mathbb{Z}(n)^X \otimes^{\mathbb{L}} \mathbb{Z}(m)^X \to \mathbb{Z}(n+m)^X$ inducing the multiplication on motivic cohomology. (These multiplications are in fact part of a graded E_∞-structure (which follows from the existence of the strong periodization, see Section C), but we do not make this explicit since we have no application for this enhanced structure.) If X is