Mémoires
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Numéro 163
Nouvelle série

LIFTING THE CARTIER TRANSFORM OF OGUS-VOLOGODSKY MODULO p^n

Daxin XU

2 0 1 9

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 35 € ($ 52)
Abonnement électronique : 113 € ($ 170)
Abonnement avec supplément papier : 167 €, hors Europe : 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat

Mémoires de la SMF
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
`memoires@smf.emath.fr` • `http://smf.emath.fr/`

ISSN papier 0249-633-X; électronique : 2275-3230

ISBN 978-2-85629-909-8

`doi:10.24033/msmf.471`

Directeur de la publication : Stéphane SEURET

MÉMOIRES DE LA SMF 163

LIFTING THE CARTIER TRANSFORM OF OGUS-VOLOGODSKY MODULO p^n

Daxin Xu

Société Mathématique de France 2019

Daxin Xu

Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA.

E-mail : daxinxu@caltech.edu

Texte reçu le 17 juillet 2017, révisé le 25 avril 2019, accepté le 16 mai 2019.

2000 ***Mathematics Subject Classification.*** **–** 14F30, 14F10.

Key words and phrases. **–** p-adic cohomology, crystalline cohomology, p-adic Hodge theory.

Mots clefs. **–** Cohomologie p-adique, cohomologie cristalline, théorie de Hodge p-adique.

LIFTING THE CARTIER TRANSFORM OF OGUS-VOLOGODSKY MODULO p^n

Daxin Xu

Abstract. – Let W be the ring of the Witt vectors of a perfect field of characteristic p, $\mathfrak{X}$ a smooth formal scheme over W, $\mathfrak{X}'$ the base change of $\mathfrak{X}$ by the Frobenius morphism of W, $\mathfrak{X}'_2$ the reduction modulo p^2 of $\mathfrak{X}'$ and X the special fiber of $\mathfrak{X}$. We lift the Cartier transform of Ogus-Vologodsky defined by $\mathfrak{X}'_2$ modulo p^n. More precisely, we construct a functor from the category of p^n-torsion $\mathscr{O}_{\mathfrak{X}'}$-modules with integrable p-connection to the category of p^n-torsion $\mathscr{O}_{\mathfrak{X}}$-modules with integrable connection, each subject to suitable nilpotence conditions. Our construction is based on Oyama's reformulation of the Cartier transform of Ogus-Vologodsky in characteristic p. If there exists a lifting $F : \mathfrak{X} \to \mathfrak{X}'$ of the relative Frobenius morphism of X, our functor is compatible with a functor constructed by Shiho from F. As an application, we give a new interpretation of Faltings' relative Fontaine modules and of the computation of their cohomology.

Résumé **(Relèvement de la transformée de Cartier d'Ogus-Vologodsky modulo p^n)**

Soient W l'anneau des vecteurs de Witt d'un corps parfait de caractéristique $p > 0$, $\mathfrak{X}$ un schéma formel lisse sur W, $\mathfrak{X}'$ le changement de base de $\mathfrak{X}$ par l'endomorphisme de Frobenius de W, $\mathfrak{X}'_2$ la réduction modulo p^2 de $\mathfrak{X}'$ et X la fibre spéciale de $\mathfrak{X}$. On relève la transformée de Cartier d'Ogus-Vologodsky définie par $\mathfrak{X}'_2$. Plus précisément, on construit un foncteur de la catégorie des $\mathscr{O}_{\mathfrak{X}'}$-modules de p^n-torsion à p-connexion intégrable dans la catégorie des $\mathscr{O}_{\mathfrak{X}}$-modules de p^n-torsion à connexion intégrable, chacune étant soumise à des conditions de nilpotence appropriées. S'il existe un relèvement $F : \mathfrak{X} \to \mathfrak{X}'$ du morphisme de Frobenius relatif de X, notre foncteur est compatible avec une construction « locale » de Shiho définie par F. Comme application de la transformée de Cartier modulo p^n, on donne une nouvelle interprétation des modules de Fontaine relatifs introduits par Faltings et du calcul de leur cohomologie.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. – In his seminal work [**34**], Simpson established a deep relation between complex representations of the fundamental group of a projective complex manifold X and Higgs modules on X, leading to a theory called *nonabelian Hodge theory*. Recall that a Higgs module on X is a coherent sheaf M together with an $\mathscr{O}_X$-linear morphism $\theta : M \to M \otimes_{\mathscr{O}_X} \Omega^1_{X/\mathbb{C}}$ such that $\theta \wedge \theta = 0$. (Simpson's result uses, but is much deeper than, the Riemann-Hilbert correspondence relating representations of the fundamental group and modules with integrable connection.) In [**14**], Faltings developed a partial p-adic analog of Simpson correspondence for p-adic local systems on varieties over p-adic fields.

On the other hand, in [**31**], Ogus and Vologodsky constructed a version of nonabelian Hodge theory in characteristic p. If X is a smooth scheme over a perfect field k of characteristic $p > 0$, they established an equivalence, called *Cartier transform*, between certain modules with integrable connection on X/k and certain Higgs modules on X/k, depending on a lifting of X' (the base change of X by the Frobenius morphism of k) to $\mathrm{W}_2(k)$. They also constructed a canonical quasi-isomorphism betweeen certain truncations of the de Rham complex of a module with integrable connection and of the Higgs complex of its Cartier transform. This result generalizes the Cartier isomorphism and the decomposition of the de Rham complex given by Deligne-Illusie [**11**]; it is also an analog of a corresponding result in Simpson's theory.

The relation between Faltings' p-adic Simpson correspondence and the Cartier transform is not yet understood. The first difficulty is to lift the Cartier transform modulo p^n. This is our main goal in the present article. Shiho [**33**] constructed a "local" lifting of the Cartier transform modulo p^n under the assumption of a lifting of the relative Frobenius morphism modulo p^{n+1}. In [**32**], Oyama gave a new construction of the Cartier transform of Ogus-Vologodsky as the inverse image by a morphism of topoi. His work is inspired by Tsuji's approach to the p-adic Simpson correspondence ([**2**] IV). In this article, we use Oyama topoi to "glue" Shiho's functor and obtain a lifting of the Cartier transform modulo p^n under the (only) assumption that X lifts to a smooth formal scheme over W.

1.2. – Shiho's construction applies to modules with λ-*connection*, a notion of introduced by Deligne. Let $f : X \to S$ be a smooth morphism of schemes, M an $\mathscr{O}_X$-module and $\lambda \in \Gamma(S, \mathscr{O}_S)$. A λ-*connection on M relative to S* is an $f^{-1}(\mathscr{O}_S)$-linear morphism $\nabla : M \to M \otimes_{\mathscr{O}_X} \Omega^1_{X/S}$ such that $\nabla(xm) = x\nabla(m) + \lambda m \otimes d(x)$ for every local sections x of $\mathscr{O}_X$ and m of M. 1-connections correspond to the classical notion of connections, and 0-connections to Higgs fields. The integrability of λ-connections is defined in the same way as for connections. We denote by $\mathrm{MIC}(X/S)$ (resp. λ-$\mathrm{MIC}(X/S)$) the category of $\mathscr{O}_X$-modules with integrable connection (resp. λ-connection) relative to S.

1.3. – In the following, if we use a gothic letter $\mathfrak{T}$ to denote an adic formal W-scheme, the corresponding roman letter T will denote its special fiber. Let $\mathfrak{X}$ be a smooth formal scheme over W and n an integer $\geqslant 1$. We denote by $\sigma : \mathrm{W} \to \mathrm{W}$ the Frobenius automorphism of W, by $\mathfrak{X}'$ the base change of $\mathfrak{X}$ by σ and by $\mathfrak{X}_n$ the reduction of $\mathfrak{X}$ modulo p^n. In [**33**], Shiho constructed a "local" lifting modulo p^n of the Cartier transform of Ogus-Vologodsky defined by $\mathfrak{X}'_2$, using a lifting $F_{n+1} : \mathfrak{X}_{n+1} \to \mathfrak{X}'_{n+1}$ of the relative Frobenius morphism $F_{X/k} : X \to X'$ of X.

The image of the differential morphism $dF_{n+1} : F^*_{n+1}(\Omega^1_{\mathfrak{X}'_{n+1}/\mathrm{W}_{n+1}}) \to \Omega^1_{\mathfrak{X}_{n+1}/\mathrm{W}_{n+1}}$ of F_{n+1} is contained in $p\Omega^1_{\mathfrak{X}_{n+1}/\mathrm{W}_{n+1}}$. Dividing by p, it induces an $\mathscr{O}_{\mathfrak{X}_n}$-linear morphism

$$dF_{n+1}/p : F^*_n(\Omega^1_{\mathfrak{X}'_n/\mathrm{W}_n}) \to \Omega^1_{\mathfrak{X}_n/\mathrm{W}_n}.$$

Shiho defined a functor (depending on F_{n+1}) ([**33**] 2.5)

$$\begin{aligned} \Phi_n : p\text{-}\mathrm{MIC}(\mathfrak{X}'_n/\mathrm{W}_n) &\to \mathrm{MIC}(\mathfrak{X}_n/\mathrm{W}_n) \\ (M', \nabla') &\mapsto (F^*_n(M'), \nabla), \end{aligned} \tag{1.3.1}$$

where $\nabla : F^*_n(M') \to \Omega^1_{\mathfrak{X}_n/\mathrm{W}_n} \otimes_{\mathscr{O}_{\mathfrak{X}_n}} F^*_n(M')$ is the integrable connection defined for every local section e of M' by

$$\nabla(F^*_n(e)) = (\mathrm{id} \otimes \frac{dF_{n+1}}{p})(F^*_n(\nabla'(e))). \tag{1.3.2}$$

Shiho showed that the functor Φ_n induces an equivalence of categories between the full subcategories of p-$\mathrm{MIC}(\mathfrak{X}'_n/\mathrm{W}_n)$ and of $\mathrm{MIC}(\mathfrak{X}_n/\mathrm{W}_n)$ consisting of quasi-nilpotent objects ([**33**] Thm. 3.1). When $n = 1$, Ogus and Vologodsky proved that the functor Φ_1 is compatible with the Cartier transform defined by $\mathfrak{X}'_2$ ([**31**] Thm. 2.11; [**33**] 1.12).

1.4. – The categories of connections and their analogs we will be studying can be understood geometrically using the language of groupoids. Our groupoids will be relatively affine and hence correspond to Hopf algebras. If $(\mathscr{T}, A)$ is a ringed topos, *a Hopf A-algebra* is the data of a ring B of $\mathscr{T}$ together with five homomorphisms

$$A \underset{d_1}{\overset{d_2}{\rightrightarrows}} B, \qquad \delta : B \to B \otimes_A B \text{ (comultiplication)},$$

$$\pi : B \to A \text{ (counit)}, \qquad \sigma : B \to B \text{ (antipode)},$$

where the tensor product $B \otimes_A B$ is taken on the left (resp. right) for the A-algebra structure of B defined by d_2 (resp. d_1), satisfying the compatibility conditions for coalgebras (cf. 4.2, **[4]** II 1.1.2).

A B-stratification on an A-module M is a B-linear isomorphism

$$\varepsilon : B \otimes_A M \xrightarrow{\sim} M \otimes_A B, \tag{1.4.1}$$

where the tensor product is taken on the left (resp. right) for the A-algebra structure defined by d_2 (resp. d_1), satisfying $\pi^*(\varepsilon) = \mathrm{id}_M$ and a cocycle condition (cf. 5.4).

1.5. – A classical example of a Hopf algebra is given by the PD-envelope of the diagonal immersion. Let $\mathfrak{X}$ be a smooth formal W-scheme, $\mathfrak{X}^2$ the product of two copies of $\mathfrak{X}$ over W. For any $n \geqslant 1$, we denote by $P_{\mathfrak{X}_n}$ the PD-envelope of the diagonal immersion $\mathfrak{X}_n \to \mathfrak{X}_n^2$ compatible with the canonical PD-structure on $(\mathrm{W}_n, p\,\mathrm{W}_n)$ and by $P_{\mathfrak{X}}$ the associated adic formal W-scheme. The $\mathscr{O}_{\mathfrak{X}}$-bialgebra $\mathscr{O}_{P_{\mathfrak{X}}}$ of $\mathfrak{X}_{\mathrm{zar}}$ is naturally equipped with a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure (i.e., for every $n \geqslant 1$, a Hopf $\mathscr{O}_{\mathfrak{X}_n}$-algebra structure on $\mathscr{O}_{P_{\mathfrak{X}_n}}$, which is compatible) (cf. 4.7, 5.10).

A quasi-nilpotent integrable connection relative to W_n on an $\mathscr{O}_{\mathfrak{X}_n}$-module M (cf. 5.3) is equivalent to an $\mathscr{O}_{P_{\mathfrak{X}}}$-stratification on M (**[5]** 4.12). Following Shiho **[33]**, we give below an analogous description of p-connections; the relevant Hopf algebra is constructed by dilatation (certain distinguished open subset of admissible blow-up) in formal geometry.

1.6. – We define by dilatation an adic formal $\mathfrak{X}^2$-scheme $R_{\mathfrak{X}}$ satisfying the following conditions (3.5).

(i) The canonical morphism $R_{\mathfrak{X},1} \to X^2$ factors through the diagonal immersion $X \to X^2$.

(ii) Let $X \to \mathfrak{X}^2$ be the morphism induced by the diagonal immersion. For any flat formal W-scheme $\mathfrak{Y}$ and any W-morphisms $f : \mathfrak{Y} \to \mathfrak{X}^2$ and $g : Y \to X$ which fit into the following commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathfrak{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & \mathfrak{X}^2, \end{array}$$

there exists a unique W-morphism $f' : \mathfrak{Y} \to R_{\mathfrak{X}}$ lifting f.

We denote abusively by $\mathscr{O}_{R_{\mathfrak{X}}}$ the direct image of $\mathscr{O}_{R_{\mathfrak{X}}}$ via the morphism $R_{\mathfrak{X},\mathrm{zar}} \to \mathfrak{X}_{\mathrm{zar}}$ (i). Using the universal property of $R_{\mathfrak{X}}$, we show that $\mathscr{O}_{R_{\mathfrak{X}}}$ is equipped with a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure (4.11).

The diagonal immersion $\mathfrak{X} \to \mathfrak{X}^2$ induces a closed immersion $\iota : \mathfrak{X} \to R_{\mathfrak{X}}$ (3.5). For any $n \geqslant 1$, we denote by $T_{\mathfrak{X},n}$ the PD-envelope of $\iota_n : \mathfrak{X}_n \to R_{\mathfrak{X},n}$ compatible with the canonical PD-structure on $(\mathrm{W}_n, p\,\mathrm{W}_n)$. The schemes $\{T_{\mathfrak{X},n}\}_{n \geqslant 1}$ form an adic inductive system and we denote by $T_{\mathfrak{X}}$ the associated adic formal W-scheme. By the universal property of PD-envelope, the formal Hopf algebra structure on $\mathscr{O}_{R_{\mathfrak{X}}}$ extends to a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure on the $\mathscr{O}_{\mathfrak{X}}$-bialgebra $\mathscr{O}_{T_{\mathfrak{X}}}$ of $\mathfrak{X}_{\mathrm{zar}}$ (5.15).

In ([**33**] Prop. 2.9), Shiho showed that for any $n \geqslant 1$ and any $\mathscr{O}_{\mathfrak{X}_n}$-module M, an $\mathscr{O}_{T_{\mathfrak{X}}}$-stratification on M is equivalent to a quasi-nilpotent integrable p-connection on M (cf. 5.17).

1.7. – Shiho's local construction deals with modules with p-connection and connection, which is different to the (global) Cartier transform of Ogus-Vologodsky. We need a fourth Hopf algebra, introduced by Oyama [**32**], and we will use it to define a notion of stratification that will enable us to globalize Shiho's construction.

For any k-scheme Y, we denote by $\underline{Y}$ the closed subscheme of Y defined by the ideal sheaf of $\mathscr{O}_Y$ consisting of the sections of $\mathscr{O}_Y$ whose pth power is zero. In (3.5), 4.9, we construct an adic formal $\mathfrak{X}^2$-scheme $Q_{\mathfrak{X}}$ satisfying the following conditions.

(i) The canonical morphism $\underline{Q_{\mathfrak{X},1}} \to X^2$ factors through the diagonal immersion $X \to X^2$.

(ii) For any flat formal W-scheme $\mathfrak{Y}$ and any W-morphisms $f : \mathfrak{Y} \to \mathfrak{X}^2$ and $g : \underline{Y} \to X$ which fit into the following commutative diagram

$$\begin{array}{ccc} \underline{Y} & \longrightarrow & \mathfrak{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & \mathfrak{X}^2, \end{array}$$

there exists a unique W-morphism $f' : \mathfrak{Y} \to Q_{\mathfrak{X}}$ lifting f.

We denote abusively by $\mathscr{O}_{Q_{\mathfrak{X}}}$ the direct image of $\mathscr{O}_{Q_{\mathfrak{X}}}$ via the morphism $Q_{\mathfrak{X},\mathrm{zar}} \to \mathfrak{X}_{\mathrm{zar}}$ (i). It is also equipped with a formal Hopf $\mathscr{O}_{\mathfrak{X}}$-algebra structure (4.11).

Let $P_{\mathfrak{X}}$ be the formal $\mathfrak{X}^2$-scheme defined in 1.5, $\iota : \mathfrak{X} \to P_{\mathfrak{X}}$ the canonical morphism lifting the diagonal immersion $\mathfrak{X} \to \mathfrak{X}^2$ and $\mathscr{J}$ the PD-ideal of $\mathscr{O}_{P_X}$ associated to ι_1. For any local section of $\mathscr{J}$, we have $x^p = p!x^{[p]} = 0$. Then we deduce a closed immersion $\underline{P_X} \to X$ over $\mathfrak{X}^2$. By the universal property of $Q_{\mathfrak{X}}$, we obtain an $\mathfrak{X}^2$-morphism $\lambda : P_{\mathfrak{X}} \to Q_{\mathfrak{X}}$.

1.8. – The groupoids and Hopf algebras constructed above give a geometric interpretation of Shiho's functor Φ and of a variation of Φ which can be globalized. Let $F : \mathfrak{X} \to \mathfrak{X}'$ be a lifting of the relative Frobenius morphism $F_{X/k}$ of X. By the universal properties of $R_{\mathfrak{X}'}$ and of PD-envelopes, the morphism $F^2 : \mathfrak{X}^2 \to \mathfrak{X}'^2$ induce morphisms $\psi : Q_{\mathfrak{X}} \to R_{\mathfrak{X}'}$ (6.6) and $\varphi : P_{\mathfrak{X}} \to T_{\mathfrak{X}'}$ (6.8) above F^2 which fit into a commutative diagram (6.9.1)

(1.8.1)
$$\begin{array}{ccc} P_{\mathfrak{X}} & \xrightarrow{\ \varphi\ } & T_{\mathfrak{X}'} \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \varpi} \\ Q_{\mathfrak{X}} & \xrightarrow{\ \psi\ } & R_{\mathfrak{X}'}, \end{array}$$

where $\varpi : T_{\mathfrak{X}'} \to R_{\mathfrak{X}'}$ (1.6) and $\lambda : P_{\mathfrak{X}} \to Q_{\mathfrak{X}}$ (1.7) are independent of F. Moreover, ψ and φ induce homomorphisms of formal Hopf algebras $\mathscr{O}_{R_{\mathfrak{X}'}} \to F_*(\mathscr{O}_{Q_{\mathfrak{X}}})$ and $\mathscr{O}_{T_{\mathfrak{X}'}} \to F_*(\mathscr{O}_{P_{\mathfrak{X}}})$. The above diagram induces a commutative diagram (6.9.2)

(1.8.2)
$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{category of } \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{O}_{R_{\mathfrak{X}'}}\text{-stratification}\end{array}\right\} & \xrightarrow{\psi_n^*} & \left\{\begin{array}{c}\text{category of } \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{O}_{Q_{\mathfrak{X}}}\text{-stratification}\end{array}\right\} \\
\Big\downarrow{\scriptstyle \varpi_n^*} & & \Big\downarrow{\scriptstyle \lambda_n^*} \\
\left\{\begin{array}{c}\text{category of } \mathscr{O}_{\mathfrak{X}'_n}\text{-modules} \\ \text{with } \mathscr{O}_{T_{\mathfrak{X}'}}\text{-stratification}\end{array}\right\} & \xrightarrow{\varphi_n^*} & \left\{\begin{array}{c}\text{category of } \mathscr{O}_{\mathfrak{X}_n}\text{-modules} \\ \text{with } \mathscr{O}_{P_{\mathfrak{X}}}\text{-stratification}\end{array}\right\}.
\end{array}$$

In ([**33**] 2.17), Shiho showed that the functor φ_n^* is compatible with the functor Φ_n defined by F (1.3.1), via the equivalence between the category of modules with quasi-nilpotent integrable connection (resp. p-connection) and the category of modules with $\mathscr{O}_{P_{\mathfrak{X}}}$-stratification (resp. $\mathscr{O}_{T_{\mathfrak{X}}}$-stratification).

1.9. – Let us explain the Oyama sites $\mathscr{E}$ and $\underline{\mathscr{E}}$ whose crystals corresponding to $\mathscr{O}_{Q_{\mathfrak{X}}}$ and $\mathscr{O}_{R_{\mathfrak{X}}}$ stratification, and a morphism of topoi which will be used to lift the Cartier transform and to globalize the funtor ψ_n^*.

Let X be a scheme over k. An object of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) is a triple $(U, \mathfrak{T}, u)$ consisting of an open subscheme U of X, a flat formal W-scheme $\mathfrak{T}$ and an affine k-morphism $u : T \to U$ (resp. $u : \underline{T} \to U$ (1.7)). Morphisms are defined in a natural way (cf. 7.1). We denote by $\mathscr{E}'$ Oyama's category associated to the k-scheme X'. We denote by $\widetilde{\mathscr{E}}$ (resp. $\widetilde{\underline{\mathscr{E}}}$) the topos of sheaves of sets on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) with respect to the Zarisiki topology (7.8).

Let $(U, \mathfrak{T}, u)$ be an object of $\underline{\mathscr{E}}$. The relative Frobenius morphism $F_{T/k} : T \to T'$ factors through a k-morphism $f_{T/k} : T \to \underline{T}'$. We have a commutative diagram

(1.9.1)
$$\begin{array}{ccccc}
U & \xleftarrow{u} & \underline{T} & \hookrightarrow & T \\
\Big\downarrow{\scriptstyle F_{U/k}} & & \Big\downarrow{\scriptstyle F_{\underline{T}/k}} & \swarrow{\scriptstyle f_{T/k}} & \Big\downarrow{\scriptstyle F_{T/k}} \\
U' & \xleftarrow{u'} & \underline{T}' & \hookrightarrow & T',
\end{array}$$

where the vertical arrows denote the relative Frobenius morphisms. Then $(U', \mathfrak{T}, u' \circ f_{T/k})$ is an object of $\mathscr{E}'$. We obtain a functor (9.1.2)

(1.9.2)
$$\rho : \underline{\mathscr{E}} \to \mathscr{E}', \qquad (U, \mathfrak{T}, u) \mapsto (U', \mathfrak{T}, u' \circ f_{T/k}).$$

The functor ρ is continuous and cocontinuous (9.3) and induces a morphism of topoi (9.1.3)

(1.9.3)
$$\mathrm{C}_{X/\mathrm{W}} : \widetilde{\underline{\mathscr{E}}} \to \widetilde{\mathscr{E}}'$$

such that its inverse image functor is induced by the composition with ρ.

1.10. – Let n be an integer $\geqslant 1$. The contravariant functor $(U, \mathfrak{T}, u) \mapsto \Gamma(\mathfrak{T}, \mathscr{O}_{\mathfrak{T}_n})$ defines a sheaf of rings on $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$) that we denote by $\mathscr{O}_{\mathscr{E},n}$ (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$). By definition, we have $\mathrm{C}^*_{X/\mathrm{W}}(\mathscr{O}_{\mathscr{E}',n}) = \mathscr{O}_{\underline{\mathscr{E}},n}$. To give an $\mathscr{O}_{\mathscr{E},n}$-module (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-module) $\mathscr{F}$ amounts to give the following data (8.2):

(i) For every object $(U, \mathfrak{T}, u)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), an $u_*(\mathscr{O}_{\mathfrak{T}_n})$-module $\mathscr{F}_{(U,\mathfrak{T})}$ of U_{zar}.

(ii) For every morphism $f : (U_1, \mathfrak{T}_1, u_1) \to (U_2, \mathfrak{T}_2, u_2)$ of $\mathscr{E}$ (resp. $\underline{\mathscr{E}}$), an $u_{1*}(\mathscr{O}_{\mathfrak{T}_{1,n}})$-linear morphism

$$c_f : u_{1*}(\mathscr{O}_{\mathfrak{T}_{1,n}}) \otimes_{(u_{2*}(\mathscr{O}_{\mathfrak{T}_{2,n}}))|_{U_1}} (\mathscr{F}_{(U_2,\mathfrak{T}_2)})|_{U_1} \to \mathscr{F}_{(U_1,\mathfrak{T}_1)},$$

satisfying a cocycle condition for the composition of morphisms as in ([**5**] 5.1).

Following ([**5**] 6.1), we say that $\mathscr{F}$ is a *crystal* if c_f is an isomorphism for every morphism f and that $\mathscr{F}$ is *quasi-coherent* if $\mathscr{F}_{(U,\mathfrak{T})}$ is a quasi-coherent $u_*(\mathscr{O}_{\mathfrak{T}_n})$-module of U_{zar} for every object $(U, \mathfrak{T}, u)$. We denote by $\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E},n})$ (resp. $\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n})$) the category of quasi-coherent crystals of $\mathscr{O}_{\mathscr{E},n}$-modules (resp. $\mathscr{O}_{\underline{\mathscr{E}},n}$-modules).

The following are the main results of this article.

Proposition 1.11 (8.10). – *Let $\mathfrak{X}$ be a smooth formal $\mathscr{S}$-scheme and X its special fiber. There exists a canonical equivalence of categories between the category $\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E},n})$ (resp. $\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n})$) and the category of quasi-coherent $\mathscr{O}_{\mathfrak{X}_n}$-modules with $\mathscr{O}_{R_{\mathfrak{X}}}$-stratification (resp. $\mathscr{O}_{Q_{\mathfrak{X}}}$-stratification)* (1.4), *1.6, 1.7.*

Theorem 1.12 (9.12). – *Let X be a smooth k-scheme. Then, for any $n \geqslant 1$, the inverse image and the direct image functors of the morphism* $\mathrm{C}_{X/\mathrm{W}}$ (1.9.3) *induce equivalences of categories quasi-inverse to each other*

(1.12.1) $$\mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\mathscr{E}',n}) \rightleftarrows \mathscr{C}^{\mathrm{qcoh}}(\mathscr{O}_{\underline{\mathscr{E}},n}).$$

The theorem is proved by fppf descent for quasi-coherent modules.

We call *Cartier equivalence modulo p^n* the equivalence of categories $\mathrm{C}^*_{X/\mathrm{W}}$ (1.12.1). Indeed, given a smooth formal W-scheme $\mathfrak{X}$ with special fiber X, Oyama proved 1.12 in the case $n = 1$ and showed that $\mathrm{C}^*_{X/\mathrm{W}}$ is compatible with the Cartier transform of Ogus-Vologodsky defined by the lifting $\mathfrak{X}'_2$ of X' (cf. [**32**] Section 1.5). In Section 12, we reprove the later result in a different way (12.22).

The following result explains the relation between the Cartier equivalence $\mathrm{C}^*_{X/\mathrm{W}}$ and Shiho's construction, in the presence of a lifting of Frobenius.

Proposition 1.13 (9.17). – *Let $\mathfrak{X}$ be a smooth formal* W*-scheme, X its special fiber, $F : \mathfrak{X} \to \mathfrak{X}'$ a lifting of the relative Frobenius morphism $F_{X/k}$ of X and ψ_n^* the functor*