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MODULI SPACES OF FLAT TORI AND ELLIPTIC HYPERGEOMETRIC FUNCTIONS

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MODULI SPACES OF FLAT TORI AND ELLIPTIC HYPERGEOMETRIC FUNCTIONS

Sélim Ghazouani, Luc Pirio

Abstract. – In the genus one case, we make explicit some constructions of Veech [**80**] on flat surfaces and generalize some geometric results of Thurston [**77**] about moduli spaces of flat spheres as well as some equivalent ones but of an analytico-cohomological nature of Deligne and Mostow [**11**], on the monodromy of Appell-Lauricella hypergeometric functions.

In the dizygotic twin paper [**20**], we follow Thurston's approach and study moduli spaces of flat tori with cone singularities and prescribed holonomy by means of geometrical methods relying on surgeries on flat surfaces. In the present memoir, we study the same objects making use of analytical and cohomological methods, more in the spirit of Deligne-Mostow's paper.

Our starting point is an explicit formula for flat metrics with cone singularities on elliptic curves, in terms of theta functions. From this, we deduce an explicit description of Veech's foliation: at the level of the Torelli space of n-marked elliptic curves, it is given by an explicit affine first integral. From the preceding result, one determines exactly which leaves of Veech's foliation are closed subvarieties of the moduli space $\mathcal{M}_{1,n}$ of n-marked elliptic curves. We also give a local explicit expression, in terms of hypergeometric elliptic integrals, for the Veech map by means of which is defined the complex hyperbolic structure of a leaf.

Then we focus on the $n = 2$ case: in this situation, Veech's foliation does not depend on the values of the cone angles of the flat tori considered. Moreover, a leaf which is a closed subvariety of $\mathcal{M}_{1,2}$ is actually algebraic and is isomorphic to a modular curve $Y_1(N)$ for a certain integer $N \geq 2$. In the considered situation, the leaves of Veech's foliation are $\mathbb{CH}^1$-curves. By specializing some results of Mano and Watanabe [**54**], we make explicit the Schwarzian differential equation satisfied by the $\mathbb{CH}^1$-developing map of any leaf and use this to prove that the metric completions of the algebraic ones are complex hyperbolic conifolds which are obtained by adding some of its cusps to $Y_1(N)$. Furthermore, we explicitly compute the conifold angle at any cusp $\mathfrak{c} \in X_1(N)$, the latter being 0 (i.e., $\mathfrak{c}$ is a usual cusp) exactly when it does not belong to the metric completion of the considered algebraic leaf.

In the last chapter of this memoir, we discuss various aspects of the objects previously considered, such as: some particular cases that we make explicit, some links with classical hypergeometric functions in the simplest cases. We explain how to explicitly compute the $\mathbb{CH}^1$-holonomy of any given algebraic leaf, which is important in order to determine when the image of such a holonomy is a lattice in $\mathrm{Aut}(\mathbb{CH}^1) \simeq \mathrm{PSL}(2, \mathbb{R})$. Finally, we compute the hyperbolic volumes of some algebraic leaves of Veech's foliation and we use this to give an explicit formula for Veech's volume of the moduli space $\mathcal{M}_{1,2}$. In particular, we show that this volume is finite, as conjectured in [**80**].

The memoir ends with two appendices. The first consists in a short and easy introduction to the notion of $\mathbb{CH}^1$-conifold. The second one is devoted to the Gauß-Manin connection associated to our problem: we first give a general and detailed abstract treatment then we consider the specific case of n-punctured elliptic curves, which is made completely explicit when $n = 2$.

Résumé **(Espaces de modules de tores plats et fonctions hypergéométriques elliptiques)**

En genre 1, nous rendons explicites certaines constructions de Veech sur les surfaces plates et généralisons des résultats géométriques de Thurston [**77**] sur les espaces de modules de sphères plates ainsi que des résultats équivalents de Deligne et Mostow [**11**], d'une nature analytico-cohomologique, qui concernent la monodromie des fonctions hypergéométriques d'Appell-Lauricella.

Dans le papier jumeau [**20**], nous reprenons l'approche de Thurston et étudions les espaces de modules de tores plats avec des singularités coniques et à l'holonomie prescrite via des méthodes géométriques obtenues au moyen d'opérations de chirurgie faites sur les surfaces plates considérées. Dans le présent mémoire, nous étudions les même objets mais en utilisant des méthodes analytiques et cohomologiques, davantage dans l'esprit de l'article de Deligne et Mostow.

Notre point de départ est une formule explicite pour les métriques plates avec des singularités coniques sur les courbes elliptiques, en termes de fonctions thêta. On en déduit une description explicite du feuilletage de Veech: au niveau de l'espace de Torelli des courbes elliptiques avec n points marqués, il est défini par une intégrale première affine explicite. Cela nous permet de déterminer exactement quelles sont les feuilles du feuilletage de Veech qui sont des sous-variétés fermées de l'espace de module $\mathcal{M}_{1,n}$ des courbes elliptiques avec n points marqués. Nous donnons aussi une expression locale explicite, en termes d'intégrales hypergéométriques elliptiques, de l'application de Veech qui permet de définir une structure hyperbolique complexe sur une feuille donnée.

On se concentre alors sur le cas $n = 2$: dans cette situation, le feuilletage de Veech ne dépend pas des valeurs des angles coniques des tores plats considérés. De plus, une feuille qui est une sous-variété fermée de $\mathcal{M}_{1,2}$ est en fait algébrique et isomorphe à une courbe modulaire $Y_1(N)$ pour un certain entier $N \geq 2$. Dans le cas particulier considéré, les feuilles du feuilletage de Veech sont des $\mathbb{CH}^1$-courbes. En spécialisant certains résultats de Mano et [**54**], nous rendons explicite l'équation différentielle Schwarzienne

que satisfait la $\mathbb{CH}^1$-développante d'une feuille et utilisons cela pour établir que les complétions métriques des feuilles algébriques sont des conifoldes hyperboliques complexes qui sont obtenues en rajoutant à $Y_1(N)$ certains de ses cusps. De plus, nous calculons explicitement l'angle conifolde en chaque cusp $\mathfrak{c} \in X_1(N)$, cet angle étant nul (i.e., $\mathfrak{c}$ est un cusp au sens ordinaire) exactement quand il n'appartient pas à la complétion métrique de la feuille algébrique considérée.

Dans le dernier chapitre de ce mémoire, nous discutons de plusieurs aspects des objets considérés auparavant, tels que: certains cas particuliers qui sont explicités encore davantage, certains liens avec les fonctions hypergéométriques classiques dans les cas les plus simples. Nous expliquons comment calculer explicitement la $\mathbb{CH}^1$-holonomie d'une feuille algébrique donnée, ce qui est important en vue de déterminer quand l'image d'une telle holonomie est un réseau de $\mathrm{Aut}(\mathbb{CH}^1) \simeq \mathrm{PSL}(2, \mathbb{R})$. Enfin, nous calculons le volume hyperbolique de certaines feuilles algébriques du feuilletage de Veech et utilisons cela pour obtenir une formule explicite pour le volume de Veech de l'espace de module $\mathcal{M}_{1,2}$. En particulier, nous montrons que ce volume est fini, comme conjecturé par Veech dans **[80]**.

Deux appendices viennent terminer ce mémoire. Le premier consiste en une introduction courte et élémentaire à la notion de $\mathbb{CH}^1$-conifolde. Le second appendice est dévolu à l'étude de la connexion de Gauß-Manin associée à notre problème: on en donne tout d'abord un traitement général détaillé avant de considérer plus spécifiquement le cas des courbes elliptiques n-épointées, cas qui est rendu complètement explicite lorsque $n = 2$.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Previous works

1.1.1. – The classical *hypergeometric series* defined for $|x|< 1$ by

$$F(a,b,c;x)=\sum_{n=0}^{+\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}x^n \tag{1}$$

together with the *hypergeometric differential equation* it satisfies

$$x(x-1)\cdot F''+\big(c-(1+a+b)x\big)\cdot F'-ab\cdot F=0 \tag{2}$$

certainly constitutes one of the most beautiful and important parts of the theory of special functions and of complex geometry of 19th century mathematics and has been studied by many generations of mathematicians since its first appearance in the work of Euler (see [**30**, Chap. I] for a historical account).

The link between the solutions of (2) and complex geometry is particularly well illustrated by the following very famous results obtained by Schwarz in [**71**]: he proved that when the parameters a, b and c are real and such that the three values $|1-c|$, $|c-a-b|$ and $|a-b|$ all are strictly less than 1, if F_1 and F_2 form a local basis of the space of solutions of (2) at a point distinct from the three singularities $0, 1$ and ∞ of the latter, then after analytic continuation, the associated (multivalued) *Schwarz's map*

$$S(a,b,c;\cdot)=\big[F_1:F_2\big]\;:\;\widetilde{\mathbb{P}^1\setminus\{0,1,\infty\}}\longrightarrow\mathbb{P}^1$$

actually takes values in $\mathbb{CH}^1\subset\mathbb{P}^1$ and induces a conformal isomorphism from the upper half-plane $\mathbb{H}\subset\mathbb{P}^1\setminus\{0,1,\infty\}$ onto a hyperbolic triangle [1]. Even if it is multi-valued, $S(a,b,c;\cdot)$ can be used to pull-back the standard complex hyperbolic structure of $\mathbb{CH}^1$ and to endow $\mathbb{P}^1\setminus\{0,1,\infty\}$ with a well-defined complete hyperbolic structure with cone singularities of angles $2\pi|1-c|$, $2\pi|c-a-b|$ and $2\pi|a-b|$ at $0, 1$ and ∞ respectively.

1. Actually, Schwarz has proved a more general result that covers not only the hyperbolic case but the Euclidean and the spherical cases as well. See e.g., [**30**, Chap.III§3.1] for a modern and clear exposition of the results of [**71**]

It has been known very early [(2)] that the following *hypergeometric integral*

$$F(x) = \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}dt$$

is a solution of (2). More generally, for any x distinct from $0, 1$ and ∞, any 1-cycle γ in $\mathbb{P}^1 \setminus \{0, 1, x, \infty\}$ and any determination of the multivalued function $t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}$ along γ, the (locally well-defined) map

(3) $$F_\gamma(x) = \int_\gamma t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}dt$$

is a solution of (2) and a basis of the space of solutions can be obtained by taking independent integration cycles (cf. **[90]** for a pleasant modern exposition of these classical results).

1.1.2. – Formula (3) leads naturally to a multi-variable generalization, first considered by Pochammer, Appell and Lauricella, then studied by Picard and his student Levavasseur (among others). We refer to [**48**, §1] for a more detailed overview of the constructions and results considered in the present subsection and in the next one.

Let $\alpha = (\alpha_i)_{i=0}^{n+2}$ be a fixed $(n+3)$-tuple of non-integer real parameters strictly bigger than -1 and such that $\sum_{i=0}^{n+2} \alpha_i = -2$, this numerical condition ensuring that precisely $n+3$ pairwise distinct singular points will be involved below. Given a $(n+3)$-tuple $x = (x_i)_{i=0}^{n+2}$ of distinct points on $\mathbb{P}^1$ and for a suitably chosen affine coordinate t, one defines a multivalued holomorphic function of t by setting

$$T_x^\alpha(t) = \prod_{i=0}^{n+2} (t - x_i)^{\alpha_i}.$$

Then, for any 1-cycle γ supported in $\mathbb{P}^1 \setminus \{x\}$ with $\{x\} = \{x_0, \ldots, x_{n+2}\}$ and any choice of a determination of $T_x^\alpha(t)$ along γ, one defines a *hypergeometric integral* as

(4) $$F_\gamma^\alpha(x) = \int_\gamma T_x^\alpha(t)dt = \int_\gamma \prod_{i=0}^{n+2} (t - x_i)^{\alpha_i} dt.$$

Since $T_x^\alpha(t)$ depends holomorphically on x and since γ does not meet any of the x_i's, F_γ^α is holomorphic as well. In fact, it is natural to normalize the integrand by considering only $(n+3)$-tuples x's normalized such that $x_0 = 0, x_1 = 1$ and $x_{n+2} = \infty$. This amounts to considering (4) as a multivalued function defined on the moduli space $\mathcal{M}_{0,n+3}$ of projective equivalence classes of $n+3$ distinct points on $\mathbb{P}^1$. As in the 1-dimensional case, it can be proved that the hypergeometric integrals (4) are solutions of a linear second-order differential system in n variables which can be seen as a

2. It seems that Legendre was the first to establish that

$$F(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}dt$$

holds true when $|x| < 1$ if a and c verify $0 < a < c$, cf. [**15**, p. 26].

multidimensional generalization of Gauß hypergeometric equation (2). Moreover, one obtains a basis of the space of solutions of this differential system by considering the (germs of) holomorphic functions $F^{\alpha}_{\gamma_0}, \ldots, F^{\alpha}_{\gamma_n}$ for some 1-cycles $\gamma_0, \ldots, \gamma_n$ forming a basis of a certain group of twisted homology.

1.1.3. – In this multidimensional context, the associated *generalized Schwarz's map* is the multivalued map

$$F^{\alpha} = \big[F^{\alpha}_{\gamma_i}\big]_{i=0}^{n} : \widetilde{\mathcal{M}_{0,n+3}} \longrightarrow \mathbb{P}^n.$$

It can be proved that the monodromy of this multivalued function on $\mathcal{M}_{0,n+3}$ leaves invariant a Hermitian form H^{α} on $\mathbb{C}^{n+1}$ whose signature is $(1, n)$ when all the α_i's are assumed to belong to the interval $]-1, 0[$.

In this case:

- F^{α} is an étale map with values into the image of $\{H^{\alpha} > 0\}$ in $\mathbb{P}^n$; in affine coordinates, this image is a complex ball hence a model of the complex hyperbolic space $\mathbb{CH}^n$;
- the monodromy group Γ^{α} of F^{α} is the image of the monodromy representation μ^{α} of the fundamental group of $\mathcal{M}_{0,n+3}$ in
$$\mathrm{PU}\big(\mathbb{C}^{n+1}, H^{\alpha}\big) \simeq \mathrm{PU}(1, n) = \mathrm{Aut}\big(\mathbb{CH}^n\big).$$

As in the classical 1-dimensional case, these results imply that there is a natural a priori non-complete complex hyperbolic structure on $\mathcal{M}_{0,n+3}$, obtained as the pull-back of the standard one of $\mathbb{CH}^n$ under the generalized Schwarz map F^{α}. We will denote by $\mathcal{M}_{0,\alpha}$ the moduli space $\mathcal{M}_{0,n+3}$ endowed with this $\mathbb{CH}^n$-structure.

Several authors (Picard, Levavasseur, Terada, Deligne-Mostow) have studied the case when the image of the monodromy $\Gamma^{\alpha} = \mathrm{Im}(\mu^{\alpha})$ is a discrete subgroup of $\mathrm{PU}(1, n)$. In this case, the metric completion of $\mathcal{M}_{0,\alpha}$ is an orbifold isomorphic to a quotient orbifold $\mathbb{CH}^n/\Gamma^{\alpha}$. Deligne and Mostow have obtained very satisfactory results on this problem: in **[11, 59]** (completed in **[60]**) they gave an arithmetic criterion on the α_i's, denoted by ΣINT, which is necessary and sufficient (up to a few known cases) to ensure that the hypergeometric monodromy group Γ^{α} is discrete. Moreover, they have determined all the α's satisfying this criterion and have obtained a list of 94 complex hyperbolic orbifolds of dimension bigger than or equal to 2 constructed via the theory of hypergeometric functions. Finally, they obtain that some of these orbifolds are non-arithmetic.

1.1.4. – In **[77]**, taking a different approach, Thurston obtains very similar results to Deligne-Mostow's. His approach is more topological and combinatorial and concerns moduli spaces of flat Euclidean structures on $\mathbb{P}^1$ with $n + 3$ cone singularities. For $x \in \mathcal{M}_{0,n+3}$, the metric $m^{\alpha}_x = |T^{\alpha}_x(t)dt|^2$ defines a flat structure on $\mathbb{P}^1$ with cone singularities at the x_i's. The bridge between the hypergeometric theory and Thurston's approach is made by the map $x \mapsto m^{\alpha}_x$ (see **[45]** where this bridge is investigated).

Using surgeries for flat structures on the sphere as well as Euclidean polygonal representations of such objects, Thurston recovers Deligne-Mostow's criterion as well as the finite list of 94 complex hyperbolic orbifold quotients. More generally, he proves that for any $\alpha = (\alpha_i)_{i=0}^{n+2} \in]-1,0[^{n+3}$ satisfying $\sum_{i=0}^{n+2} \alpha_i = -2$ (3) and not only for the (necessarily rational) ones satisfying ΣINT, the metric completion $\overline{\mathcal{M}}_{0,\alpha}$ carries a complex hyperbolic conifold structure (see **[77, 57]** or **[20]** for this notion) which extends the $\mathbb{CH}^n$-structure of the moduli space $\mathcal{M}_{0,\alpha}$.

1.1.5. – In the very interesting (but long and hard-reading hence not so well-known) paper **[80]**, Veech generalizes some parts of the preceding constructions by Deligne-Mostow and Thurston, the latter corresponding to the genus 0 case, to Riemann surfaces of arbitrary genus g. All of Veech's results considered below are discussed and properly stated in the Introduction of **[80]**, to which we refer the reader. The third section of **[20]** may be a handy reference as well.

Veech's starting point is a nice result by Troyanov **[78]** asserting that for any $\alpha = (\alpha_i)_{i=1}^{n} \in]-1,\infty[^n$ such that

$$\sum_{i=1}^{n} \alpha_i = 2g - 2 \tag{5}$$

and any genus g Riemann surface X with a n-tuple $x = (x_i)_{i=1}^n$ of marked distinct points on it, there exists a unique flat metric $m^\alpha_{X,x}$ of area 1 on X with cone singularities of angle $\theta_i = 2\pi(1+\alpha_i) > 0$ at x_i for every $i = 1,\dots,n$, in the conformal class associated to the complex structure of X. Equality (5) has to be assumed because, thanks to the generalization to flat surfaces with cone singularities of the Gauß-Bonnet theorem (see **[78]**), any flat structure with cone singularities has to satisfy it.
From this, Veech obtains a real analytic isomorphism

$$\begin{aligned} \mathscr{T}\!eich_{g,n} &\simeq \mathscr{E}^\alpha_{g,n} \\ \big[(X,x)\big] &\mapsto \Big[\big(X, m^\alpha_{X,x}\big)\Big] \end{aligned} \tag{6}$$

between the Teichmüller space $\mathscr{T}\!eich_{g,n}$ of n-marked Riemann surfaces of genus g and the space $\mathscr{E}^\alpha_{g,n}$ of (isotopy classes of) flat Euclidean structures with n cone points of angles $\theta_1,\dots,\theta_n$ on the marked surfaces of the same type.

Using (6) to identify the Teichmüller space with $\mathscr{E}^\alpha_{g,n}$, Veech constructs a real-analytic map

$$H^\alpha_{g,n} : \mathscr{T}\!eich_{g,n} \longrightarrow \mathbb{U}^{2g} \tag{7}$$

which associates to (the isotopy class of) a n-marked Riemann surface (X,x) of genus g the unitary linear holonomy (4) of the flat structure on X induced by $m^\alpha_{X,x}$.

3. In the realm of flat surfaces (here of genus 0) with cone singularities, this condition corresponds to the (generalization of the) Gauß-Bonnet Theorem (see § 2.7.2.1 further).

4. See § 2.7.2.1 further for the notion of '*linear holonomy*' of a flat surface.

The map (7) is a submersion and even though it is just real-analytic, Veech proves that any level set
$$\mathscr{F}_\rho^\alpha = \big(H_{g,n}^\alpha\big)^{-1}(\rho)$$
is a complex submanifold of $\mathscr{T}eich_{g,n}$ of complex dimension $2g-3+n$ if $\rho \in \mathrm{Im}(H_{g,n}^\alpha)$ is not trivial [5]. For such a unitary character ρ and under the assumption that none of the α_i's is an integer, Veech introduces a certain first cohomology space $\mathcal{H}_\rho^1$ which essentially encodes the translation parts of the Euclidean holonomies of the elements of $\mathscr{F}_\rho^\alpha$ viewed as isomorphism classes of flat surfaces (see § 4.4.1 for details). By this means, he constructs a 'complete holonomy map'
$$\mathrm{Hol}_\rho^\alpha : \mathscr{F}_\rho^\alpha \longrightarrow \mathbb{P}\mathcal{H}_\rho^1 \simeq \mathbb{P}^{2g-3+n}$$
and proves first that this map is a local biholomorphism, then that there is a Hermitian form H_ρ^α on $\mathcal{H}_\rho^1$ and that Hol_ρ^α maps $\mathscr{F}_\rho^\alpha$ into the projectivization $X_\rho^\alpha \subset \mathbb{P}^{2g-3+n}$ of the set $\{H_\rho^\alpha > 0\} \subset \mathcal{H}_\rho^1$ (compare with § 1.1.3).

By a long calculation, Veech determines explicitly the signature (p,q) of H_ρ^α and shows that it does depend only on α. The most interesting case is when $(p,q) = (1, 2g-3+n)$. Indeed, in this case Hol_ρ^α takes its values into $X_\rho^\alpha \simeq \mathbb{CH}^{2g-3+n}$ which is a Hermitian symmetric space, a Riemannian manifold in particular. By pull-back under Hol_ρ^α which is étale, one endows the leaf $\mathscr{F}_\rho^\alpha$ with a natural complex hyperbolic structure.

One occurrence of this situation is when $g = 0$ and all the α_i's belong to the interval $]-1,0[$: in this case there is only one leaf which is the whole Teichmüller space $\mathscr{T}eich_{0,n}$ and as mentioned above, one recovers precisely the case studied by Deligne-Mostow and Thurston.

1.1.6. – In addition to the genus 0 case, Veech shows that the complex hyperbolic situation also occurs in another case, namely when

(8) $\quad g = 1$ *and* *all the* α_i*'s are in* $]-1,0[$ *except one which lies in* $]0,1[$.

In this case, the level-sets $\mathscr{F}_\rho^\alpha$'s of the holonomy map $H_{g,n}^\alpha$ form a real-analytic foliation $\mathscr{F}^\alpha$ of $\mathscr{T}eich_{1,n}$ whose leaves carry natural $\mathbb{CH}^{n-1}$-structures.

A remarkable fact established by Veech [**80**, Thm. 0.7] is that the pure mapping class group $\mathrm{PMCG}_{1,n}$ leaves this foliation invariant (in the sense that the action maps leaves onto leaves, possibly permuting them) and induces biholomorphisms between the leaves which preserve their respective complex hyperbolic structure (see [**80**, Thm. 0.9]). Consequently, all the previous constructions pass to the quotient by $\mathrm{PMCG}_{1,n}$. One finally obtains a foliation, denoted by $\mathcal{F}^\alpha$, on the quotient moduli

5. Note that a necessary condition for the trivial character *1* to belong to the image of $H_{g,n}^\alpha$ is that all the α_i's are integers. In this text, we will always assume that it is not the case. However, it is worth mentioning that the case when $1 \in \mathrm{Im}(H_{g,n}^\alpha)$ is very interesting: in this case, the associated level-set $\mathscr{F}_1^\alpha$ corresponds to a strata of abelian differentials (on Riemann surfaces of genus g and with n zeros) and such objects have been the subject of many important works in recent years.