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CONSTRUCTIVE AND DESTRUCTIVE INTERFERENCES IN NONLINEAR HYPERBOLIC EQUATIONS

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CONSTRUCTIVE AND DESTRUCTIVE INTERFERENCES IN NONLINEAR HYPERBOLIC EQUATIONS

Rémi Carles, Christophe Cheverry

Abstract. – This article introduces a physically realistic model for explaining how electromagnetic waves can be internally generated, propagate and interact in strongly magnetized plasmas or in nuclear magnetic resonance experiments. It studies high frequency solutions of nonlinear hyperbolic equations for time scales at which dispersive and nonlinear effects can be present in the leading term of the solutions. It explains how the produced waves can accumulate during long times to produce constructive and destructive interferences which, in the above contexts, are part of turbulent effects.

Résumé **(Interférences constructives et destructives pour des équations hyperboliques non linéaires)**

Cet article introduit un modèle physiquement réaliste qui explique comment, dans des plasmas fortement magnétisés ou lors d'expériences de résonance magnétique nucléaire, des ondes électromagnétiques peuvent être créées, se propager et interagir. Il étudie des solutions haute fréquence de systèmes hyperboliques non linéaires pour lesquelles des effets dispersifs et non linéaires sont impliqués à l'ordre principal. Il explique les modalités selon lesquelles les ondes produites peuvent s'accumuler dans le temps long pour produire des interférences constructives et destructives qui, dans ce contexte, peuvent être interprétés comme des phénomènes de turbulence.

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CHAPTER 1

INTRODUCTION

In this introduction, we present the main aspects of our text. In Section 1.1, we introduce a simple ODE model that is intended to serve as a guideline. In Section 1.2, we extend this model to better incorporate important specificities of two realistic situations which are related to strongly magnetized plasmas (SMP) and nuclear magnetic resonance (NMR). In Section 1.3, we state under simplified assumptions our two main results, Theorems 1.3 and 1.4. We also give an overview of our article.

1.1. A toy model

Introduce the *phase* $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(t) := t + \gamma(\cos t - 1), \qquad \gamma \in]0, 1/4[. \tag{1.1}$$

Let $\varepsilon \in]0, 1]$ be a small parameter, and $\lambda \in \mathbb{C}$. Fix numbers $(j_1, j_2, \nu) \in \mathbb{N}^2 \times \mathbb{R}$ such that $j_1 + j_2 \geq 2$. Select $n \in \mathbb{Z}$ and $\omega \in \mathbb{R}$. Then, define

$$F_L(\varepsilon, t) := \varepsilon^{3/2} e^{in\varphi(t)/\varepsilon}, \qquad F_{NL}(\varepsilon, t, u) := \lambda \varepsilon^{\nu} e^{i\omega t/\varepsilon} u^{j_1} \bar{u}^{j_2}. \tag{1.2}$$

Definition 1.1. – *The number* $\mathfrak{g} := \omega + j_1 - j_2 \in \mathbb{R}$ *is called the gauge parameter associated with* F_{NL}.

Consider the ordinary differential equation on the complex plane $\mathbb{C}$ given by

$$\frac{d}{dt}u - \frac{i}{\varepsilon}u = F(\varepsilon, t, u) := F_L(\varepsilon, t) + F_{NL}(\varepsilon, t, u), \qquad u_{|t=0} = 0. \tag{1.3}$$

We can study the equation (1.3) on three different time scales:

- *Fast*, when $t \sim \varepsilon$, that is when F undergoes a few number of oscillations;
- *Normal*, when $t \sim 1$, that is when F generates $\mathcal{O}(\varepsilon^{-1})$ oscillations, whereas the periodic part $(\cos t)$ inside φ sees a few number of oscillations;
- *Slow*, when $t \sim \varepsilon^{-1}$ or $T := \varepsilon t \sim 1$, that is when F involves $\mathcal{O}(\varepsilon^{-2})$ oscillations.

In this subsection, we analyze (1.3) during long times $t \sim \varepsilon^{-1}$ or $T \sim 1$. With this in mind, we can change u according to

$$u(t) = \varepsilon e^{it/\varepsilon} \mathcal{U}(\varepsilon t), \qquad \mathcal{U}(T) := \varepsilon^{-1} e^{-iT/\varepsilon^2} u(\varepsilon^{-1} T). \tag{1.4}$$

Expressed in terms of $\mathcal{U}$, the equation (1.3) becomes

$$\frac{d}{dT}\mathcal{U} = \frac{1}{\sqrt{\varepsilon}} e^{i(n-1)T/\varepsilon^2 + in\gamma(\cos(T/\varepsilon)-1)/\varepsilon} + \lambda \varepsilon^{\nu+j_1+j_2-2} e^{i(\mathfrak{g}-1)T/\varepsilon^2} \mathcal{U}^{j_1} \bar{\mathcal{U}}^{j_2}. \tag{1.5}$$

The initial data is still zero. Denote by $\mathcal{U}_{\text{lin}}$ the solution corresponding to the linear evolution, that is the solution obtained from (1.5) when $\lambda = 0$. When $\lambda \neq 0$ and when $\nu + j_1 + j_2 > 2$, the solution to (1.5) looks like $\mathcal{U}_{\text{lin}}$. Our aim is to first study the expression $\mathcal{U}_{\text{lin}}$. Then, we incorporate nonlinear effects by looking at a critical size for the nonlinearity, corresponding to the special case $\lambda \neq 0$ and $\nu + j_1 + j_2 = 2$. This means to single out the following equation

$$\frac{d}{dT}\mathcal{U} = \frac{1}{\sqrt{\varepsilon}} e^{i(n-1)T/\varepsilon^2 + in\gamma(\cos(T/\varepsilon)-1)/\varepsilon} + \lambda e^{i(\mathfrak{g}-1)T/\varepsilon^2} \mathcal{U}^{j_1} \bar{\mathcal{U}}^{j_2}, \quad \mathcal{U}|_{T=0} = 0. \tag{1.6}$$

The integral formulation of (1.6) reads

$$\mathcal{U}(T) = \mathcal{U}_{\text{lin}}(T) + \lambda \int_0^T e^{i(\mathfrak{g}-1)s/\varepsilon^2} \mathcal{U}(s)^{j_1} \bar{\mathcal{U}}(s)^{j_2} ds. \tag{1.7}$$

In Paragraph 1.1.1, we first show that $\mathcal{U}_{\text{lin}}(T) = \mathcal{O}(1)$, an estimate which is sharp when $n = 1$. As a consequence, the nonlinear contribution brought by the integral term inside (1.7) is likely to be of the same order of magnitude as the linear one. It can be expected that $\mathcal{U}(T) \not\equiv \mathcal{U}_{\text{lin}}(T) + o(1)$. In Paragraph 1.1.2, we prove that this is indeed the case if and only if $\mathfrak{g} = 1$.

1.1.1. The linear case. – By construction, we have

$$u_{\text{lin}}(t) := \varepsilon e^{it/\varepsilon} \mathcal{U}_{\text{lin}}(\varepsilon t) = \varepsilon^{3/2} e^{it/\varepsilon} \int_0^t e^{i[n\varphi(s)-s]/\varepsilon} ds. \tag{1.8}$$

We start the analysis of (1.7) by looking at the part $\mathcal{U}_{\text{lin}}$ through the expression u_{lin} of (1.8). Examine the right hand side of (1.8). For harmonics $n \in \mathbb{Z}$ with $n \neq 1$, since $0 < \gamma < 1/4$, remark that

$$\forall s \in \mathbb{R}, \qquad 1/2 \leq |n\varphi'(s) - 1| = |n - 1 - \gamma n \sin s|. \tag{1.9}$$

Exploiting (1.9), a single integration by parts yields

$$\forall t \geq 0, \qquad u_{\text{lin}}(t) = \mathcal{O}\big(\varepsilon^{5/2}(1+t)\big).$$

In other words, assuming that $n \neq 1$, we find

$$\forall T \geq 0, \qquad \mathcal{U}_{\text{lin}}(T) = \mathcal{O}(\varepsilon^{3/2} + \sqrt{\varepsilon} T). \tag{1.10}$$

The situation is completely different when $n = 1$. Fix an integer $K \geq 1$. The solution u_{lin} computed at the time $t = 2K\pi$ can be viewed as a sum of contributions produced over time by the source term, namely

$$u_{\text{lin}}(2K\pi) = \sum_{k=0}^{K-1} u_k, \qquad u_k := \varepsilon^{3/2} e^{i2K\pi/\varepsilon} \int_{2k\pi}^{2(k+1)\pi} e^{i[\varphi(s)-s]/\varepsilon} ds. \tag{1.11}$$

Since the function $s \mapsto \varphi(s) - s = \gamma(\cos s - 1)$ is periodic of period 2π, the wave packets u_k can be interpreted according to $u_k = \varepsilon^{3/2} e^{i2K\pi/\varepsilon} \mathrm{v}_k$ with

$$\mathrm{v}_k = \int_{2k\pi-\pi/2}^{2k\pi+3\pi/2} e^{i\gamma(\cos s-1)/\varepsilon} ds = \mathrm{v} := \int_{-\pi/2}^{3\pi/2} e^{i\gamma(\cos s-1)/\varepsilon} ds. \tag{1.12}$$

The function $s \mapsto \gamma(\cos s - 1)$ has exactly two non-degenerate stationary points in the interval $[2k\pi - \pi/2, 2k\pi + 3\pi/2]$, at the positions $s = 2k\pi$ and $s = 2k\pi + \pi$. Using the periodicity to get rid of the boundary terms and applying stationary phase formula, it follows that

$$\mathrm{v} = \sqrt{\frac{2\pi\varepsilon}{\gamma}} e^{-i\frac{\gamma}{\varepsilon}} \left(e^{i(\frac{\gamma}{\varepsilon} - \frac{\pi}{4})} + e^{-i(\frac{\gamma}{\varepsilon} - \frac{\pi}{4})} \right) + \mathcal{O}\big(\varepsilon^{3/2}\big). \tag{1.13}$$

Let $A_\varepsilon \in \mathbb{C}$ be such that

$$A_\varepsilon^2 = \sqrt{\frac{2}{\pi\gamma}} e^{-i\frac{\gamma}{\varepsilon}} \cos\Big(\frac{\gamma}{\varepsilon} - \frac{\pi}{4}\Big), \qquad \limsup_{\varepsilon \to 0} |A_\varepsilon^2| = \sqrt{\frac{2}{\pi\gamma}} \neq 0. \tag{1.14}$$

Observe that

$$\mathrm{v} = 2\pi A_\varepsilon^2 \sqrt{\varepsilon} + \mathcal{O}\big(\varepsilon^{3/2}\big), \qquad |u_k| = 2\pi |A_\varepsilon^2| \varepsilon^2 + \mathcal{O}\big(\varepsilon^3\big). \tag{1.15}$$

The combination of (1.11), (1.14) and (1.15) indicates that, when $n = 1$, wave packets u_k of amplitude ε^2 are repeatedly created over time when solving (1.3) in the case $\lambda = 0$.

Look at (1.11). The emitted signals u_k (one per period 2π) have cumulative effects up to the stopping time $2K\pi$. They give rise to a growth rate with respect to the time variable t. For long times $T \sim 1$, assuming that $n = 1$, we can assert that

$$\mathcal{U}_{\mathrm{lin}}(T) = A_\varepsilon^2 T + \mathcal{O}(\varepsilon) = A_\varepsilon^2 \int_0^{+\infty} 1_{[0,T]}(s) ds + \mathcal{O}(\varepsilon) = \mathcal{O}(1). \tag{1.16}$$

This short discussion about the linear situation ($\lambda = 0$) highlights a difference between the cases $n \neq 1$—see (1.10)—and $n = 1$—see (1.16). This observation is important in the perspective of nonlinear effects. As a matter of fact, it allows a first selection between the different modes $n \in \mathbb{Z}$.

1.1.2. Nonlinear effects. – Here, we consider the nonlinear framework, when $\lambda \neq 0$ and $\nu + j_1 + j_2 = 2$. The difference $\mathcal{W} := \mathcal{U} - \mathcal{U}_{\mathrm{lin}}$ is subject to

$$\mathcal{W}(T) = \lambda \int_0^T e^{i(\mathfrak{g}-1)s/\varepsilon^2} (\mathcal{U}_{\mathrm{lin}} + \mathcal{W})(s)^{j_1} (\bar{\mathcal{U}}_{\mathrm{lin}} + \bar{\mathcal{W}})(s)^{j_2} ds. \tag{1.17}$$

Using a Picard scheme, it is easy to infer that the life span of the solution $\mathcal{W}$ to the integral equation (1.17), and therefore of the solution $\mathcal{U}$ to (1.6), can be bounded below by a positive constant not depending on $\varepsilon \in]0,1]$. Knowing (1.10) and (1.16), it is also possible to deduce that $\mathcal{W}(T)$ is of size $\mathcal{O}(\varepsilon^{(j_1+j_2)/2}) = \mathcal{O}(\varepsilon)$ when $n \neq 1$, and of size $\mathcal{O}(1)$ when $n = 1$. This means that the preceding dichotomy between the two cases $n \neq 1$ and $n = 1$ remains when $\lambda \neq 0$.

FACT 1. – When solving (1.6), the harmonic $n = 1$ stands out from the others. Given $T > 0$, we find $\mathcal{U}(T) = \mathcal{O}(\sqrt{\varepsilon})$ when $n \neq 1$, and $\mathcal{U}(T) = \mathcal{O}(1)$ when $n = 1$.

Assume that $\mathfrak{g} \neq 1$. The identity (1.7) becomes after an integration by parts

$$\begin{aligned}\mathcal{U}(T) = \mathcal{U}_{\mathrm{lin}}(T) &- \frac{i\lambda\varepsilon^2}{\mathfrak{g}-1} e^{i(\mathfrak{g}-1)T/\varepsilon^2} \mathcal{U}(T)^{j_1} \bar{\mathcal{U}}(T)^{j_2} \\ &+ \frac{i\lambda\varepsilon^2}{\mathfrak{g}-1} \int_0^T e^{i(\mathfrak{g}-1)s/\varepsilon^2} \partial_s \big(\mathcal{U}(s)^{j_1} \bar{\mathcal{U}}(s)^{j_2}\big) ds.\end{aligned} \tag{1.18}$$

From the Equation (1.6), since we have seen that the solution $\mathcal{U}$ is (at least) bounded, we know that $\partial_s \mathcal{U}(s) = \mathcal{O}(\varepsilon^{-1/2})$. From (1.18), it follows that

$$\forall T \in \mathbb{R}, \qquad \mathcal{U}(T) = \mathcal{U}_{\mathrm{lin}}(T) + \mathcal{O}(\varepsilon^{3/2}).$$

Now, assume that $n = 1$ and moreover that $\mathfrak{g} = 1$. To show that, in this situation, nonlinear effects actually occur, it suffices to produce an example. To this end, take $(j_1, j_2, \nu) = (2, 0, 0)$ and $\omega = -1$, so that $\mathfrak{g} = 1$. Choose $\lambda = 1$. Then, using (1.16), the identity (1.7) becomes

$$\mathcal{U}(T) = A_\varepsilon^2 T + \mathcal{O}(\varepsilon) + \int_0^T \mathcal{U}(s)^2 ds. \tag{1.19}$$

This implies that $\mathcal{U}(T) = A_\varepsilon \tan(A_\varepsilon T) + \mathcal{O}(\varepsilon)$, and therefore

$$\mathcal{U}(T) - \mathcal{U}_{\mathrm{lin}}(T) = A_\varepsilon \tan(A_\varepsilon T) - A_\varepsilon^2 T + \mathcal{O}(\varepsilon) \neq o(1).$$

In view of the above formula, the asymptotic behavior of the nonlinear solution $\mathcal{U}$ can strongly differ from the one of the linear solution $\mathcal{U}_{\mathrm{lin}}$.

FACT 2. – When solving (1.6), the gauge parameter $\mathfrak{g} = 1$ stands out from the others. When $\mathfrak{g} \neq 1$, the asymptotic behaviors of $\mathcal{U}$ and $\mathcal{U}_{\mathrm{lin}}$ when ε goes to 0 are the same. On the contrary, when $\mathfrak{g} = 1$, nonlinear effects can be expected at leading order.

1.2. A more realistic model

The preceding features, Facts 1 and 2, which have been emphasized in the case of ODEs, are still present when dealing with partial differential equations arising in strongly magnetized plasmas (SMP) or in nuclear magnetic resonance experiments (NMR). But, there are two emerging issues: the first is due to dispersive effects which are completely absent in the ODE case; the second comes from the occurrence of non-trivial spatial variations when dealing with the phase φ. At all events, the discussion becomes much more subtle, and new important phenomena can and do occur.

In order to investigate SMP or NMR, we must consider the PDE counterpart of (1.3), which is

$$\partial_t u - \frac{i}{\varepsilon} p(\varepsilon D_x) u = F = F_L + F_{NL}, \qquad u|_{t=0} = 0, \qquad 0 < \varepsilon \ll 1, \tag{1.20}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}$. The state variable is $u \in \mathbb{R}$ and $D_x := -i\partial_x$. The action of the pseudo-differential operator $p(\varepsilon D_x)$ is given on the Fourier side by the multiplier $p(\varepsilon\xi)$.

In what follows, we will focus on the scalar wave equation (1.20). The origin of equation (1.20), its physical significances and the reasons why it may be seen as a universal problem (when dealing with systems of hyperbolic equations) will be clearly explained in Chapters 2 and 3. We will work in space dimension one. The possible multidimensional effects will not be investigated here.

We now fix some notations and we introduce simplified assumptions intended to facilitate the presentation of our main results. We suppose that the symbol p is smooth, say $p \in \mathcal{C}^\infty(\mathbb{R})$. The function p is even. It is such that $p_{|[-\xi_c,\xi_c]} \equiv 0$ for some $\xi_c \geq 0$. It is strictly increasing on (ξ_c, ∞). Moreover, for large values of ξ, it is subject to

$$\text{(1.21)} \qquad \lim_{\xi\to+\infty} p(\xi) = 1, \quad \lim_{\xi\to+\infty} p'(\xi) = 0, \quad \exists \ell < 0, \quad \lim_{\xi\to+\infty} \xi^4 p''(\xi) = \ell,$$

as well as

$$\text{(1.22)} \qquad \exists D \geq 4; \qquad \forall n \in \{2,\dots,D\}, \quad \limsup_{\xi\to+\infty} \frac{|p^{(n)}(\xi)|}{p'(\xi)} < +\infty.$$

Fix some $M \in \mathbb{N}^*$. The source term F_L is defined by

$$\text{(1.23)} \qquad F_L(\varepsilon,t,x) = -\varepsilon^{3/2} \sum_{m\in[-M,M]\setminus\{0\}} a_m(\varepsilon t,t,x) e^{im\varphi(t,x)/\varepsilon}.$$

In the above line (1.23), the amplitudes $a_m(T,t,x)$ are chosen in the set $\mathcal{C}_b^\infty(\mathbb{R}^3)$ of smooth functions whose derivatives are all bounded. They are selected in such a way that, for some $\mathcal{T} > 0$ and some $r \in \mathbb{R}_+^*$ with $r < \gamma/2$, we have

$$\text{(1.24)} \qquad \forall m \in [-M,M] \setminus \{0\}, \qquad \operatorname{supp} a_m \subset]-\infty, \mathcal{T}] \times [1,+\infty[\times [-r,r].$$

The amplitude $a_1(T,t,x)$ is chosen periodic for large times in the second variable. In other words, there exists $\mathrm{t}_s \in \mathbb{R}_+^*$ and a smooth function $\underline{a}(T,t,x)$ such that

$$\text{(1.25)} \qquad \forall t \geq \mathrm{t}_s, \quad \forall n \in \mathbb{N}, \quad a_1(\cdot,t+n\pi,\cdot) \equiv \underline{a}(\cdot,t+n\pi,\cdot) \equiv \underline{a}(\cdot,t,\cdot).$$

The phase φ arising in (1.23) is more general than in (1.1). It does depend on the spatial variable $x \in \mathbb{R}$. It is the sum of a quadratic part (in t and x) and a periodic part (in t).

ASSUMPTION 1.2 (Selection of a relevant phase φ). – *The function φ is*

$$\text{(1.26)} \qquad \varphi(t,x) = t - xt + \gamma(\cos t - 1), \qquad 0 < \gamma < 1/4.$$

In Chapter 2, the above assumptions on p and φ will be motivated by the study of two realistic situations which are related to strongly magnetized plasmas (SMP) and nuclear magnetic resonance (NMR). In Chapter 3, to better incorporate important specificities of SMP and NMR, they will be somewhat generalized.

In the right hand side of (1.20), the nonlinear part F_{NL} is, up to some localization in time and space, of the same form as in the previous subsection. Select a nonnegative cut-off function χ which is equal to 1 in a neighborhood of the origin and which is such that $\operatorname{supp}\chi \subset [-1,1]$. Fix some parameter $\iota \in [0,1]$ which is aimed to measure the strength of the spatial localization. We impose

$$F_{NL}(\varepsilon,t,x,u) = \lambda\varepsilon^{\nu}\chi\Big(3-2\frac{\varepsilon t}{\mathcal{T}}\Big)\chi\left(\frac{x}{r\varepsilon^{\iota}}\right)e^{i\omega t/\varepsilon}u^{j_1}\bar{u}^{j_2}. \tag{1.27}$$

Taking into account the conditions on the support of the a_m's and χ, the term F_{NL} becomes effective only for $t \geq \mathcal{T}/\varepsilon$, that is after the term F_L has played its part. So we observe successively two distinct phenomena: a possible linear amplification, and then nonlinear interactions.

The solution u to (1.20) exists on a time interval $[0,\tilde{\mathcal{T}}/\varepsilon]$ with $\mathcal{T} < \tilde{\mathcal{T}}$. The argument is similar to the one given for the toy model. Through the change (1.4), we can reformulate the equation (1.20) in terms of $\mathcal{W} = \mathcal{U} - \mathcal{U}_{\text{lin}}$, see (5.3) and (5.4). When $\nu + j_1 + j_2 > 2$, the lifespan expressed in terms of $T = \varepsilon t$ does not shrink to $\mathcal{T}$ when ε goes to zero. Note however that, due to the quadratic nonlinearity, the global-in-time existence is not at all guaranteed concerning (1.20).

We still denote by u_{lin} the linear solution obtained from (1.20) when $\lambda = 0$. One point should be underlined here. Our discussion of the linear situation is based on the analysis in L^{∞} of oscillatory integrals appearing in a suitable wave packet decomposition of u_{lin}. The precise structure of these wave packets is lost under the influence of nonlinearities. It follows that our key argument cannot be iterated to obtain the existence and the asymptotic behavior of the solution to the full nonlinear Equation (1.20). For this reason, we do not work with (1.20). Instead, we look at the first two iterates of an associated Picard iterative scheme, which are

$$\partial_t u^{(0)} - \frac{i}{\varepsilon}p(\varepsilon D_x)u^{(0)} = F_L, \qquad u|_{t=0}^{(0)} = 0, \tag{1.28a}$$

$$\partial_t u^{(1)} - \frac{i}{\varepsilon}p(\varepsilon D_x)u^{(1)} = F_L + F_{NL}\big(u^{(0)}\big), \qquad u|_{t=0}^{(1)} = 0. \tag{1.28b}$$

Generalizing (1.4), we can define

$$\mathcal{U}^{(j)}(T,z) := \frac{1}{\varepsilon}e^{-iT/\varepsilon^2}u^{(j)}\Big(\frac{T}{\varepsilon},\varepsilon z\Big), \quad u^{(j)}(t,x) := \varepsilon e^{it/\varepsilon}\mathcal{U}^{(j)}\Big(\varepsilon t,\frac{x}{\varepsilon}\Big). \tag{1.29}$$

The expression $\mathcal{U}^{(0)}$ is the solution to the linear equation ($\lambda = 0$). Thus, we have

$$\mathcal{U}^{(0)}(T,z) = \mathcal{U}_{\text{lin}}(T,z) := \frac{1}{\varepsilon}e^{-iT/\varepsilon^2}u_{\text{lin}}\Big(\frac{T}{\varepsilon},\varepsilon z\Big).$$

Symbols like p appear when looking at special branches $\mathcal{V}$ of *characteristic varieties* describing the propagation of electromagnetic waves

$$\mathcal{V} := \big\{(t,x,\tau,\xi);\tau = p(\xi),(t,x,\xi)\in\mathbb{R}^3\big\} \subset T^*(\mathbb{R}^2) \equiv \mathbb{R}^2\times\mathbb{R}^2. \tag{1.30}$$

On the other hand, the phase φ may reflect the transport properties of particles. The graph $\mathcal{G}$ of the gradient of φ is associated with the *Lagrangian manifold*

$$\mathcal{G} := \big\{\big(t,x,\partial_t\varphi(t,x),\partial_x\varphi(t,x)\big);(t,x)\in\mathbb{R}^2\big\} \subset T^*(\mathbb{R}^2) \equiv \mathbb{R}^2\times\mathbb{R}^2. \tag{1.31}$$

In the ODE framework of Paragraph 1.1, we simply find

$$\mathcal{V}_{\text{ode}} = \big\{(t,x,1,\xi);(t,x,\xi)\in\mathbb{R}^3\big\}, \qquad \mathcal{G}_{\text{ode}} = \big\{(t,x,1-\gamma\sin t,0);(t,x)\in\mathbb{R}^2\big\},$$

so that

$$\mathcal{V}_{\rm ode} \cap \mathcal{G}_{\rm ode} = \big\{ (k\pi, x, 1, 0) ; (k,x) \in \mathbb{Z} \times \mathbb{R} \big\}. \tag{1.32}$$

Thus, the production at the successive times $k\pi$ with $k \in \mathbb{N}$ of the wave packets u_k which appear at the level of (1.11) can be interpreted as coming from positions which are inside $\mathcal{V}_{\rm ode} \cap \mathcal{G}_{\rm ode}$. This principle is illustrated in Figure 1 below, given at x fixed and $\xi = 0$, with t in abscissa and the time frequency τ in ordinate.

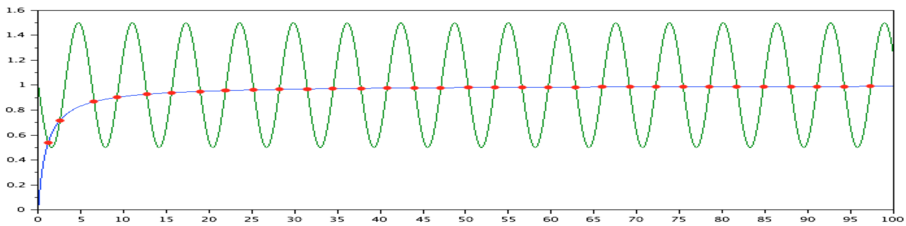

FIGURE 1. Intersection (in red) of $\mathcal{V}_{\rm ode}$ (in blue) and $\mathcal{G}_{\rm ode}$ (in green)

Similarly, in the general framework (1.20), *two-dimensional* oscillating waves u_k can emanate from the more complicated intersection

$$\mathcal{V} \cap \mathcal{G} = \big\{ \big(t, x, p(-t), -t\big) ; (t,x) \in \mathbb{R}^2 \ \text{ and } \ p(-t) = p(t) = 1 - x - \gamma \sin t \big\}.$$

In view of (1.21), for large values of $|\xi|$, the dispersion relation $p(\xi) = \tau$ mimics the choice $p \equiv 1$ of (1.3). As in (1.32), the set $\mathcal{V} \cap \mathcal{G}$ contains (near $x = 0$ and for t large enough) an infinite number of curve portions (in $\mathbb{R}^2$) which appear repeatedly in time, and from which oscillating waves u_k may be triggered.

In the framework of SMP and NMR, the symbol p and the phase φ are issued from different physical laws. They are originally unrelated, see Chapter 2. But they are connected when solving the equation (1.20). The interactions between "waves" (associated with p) and "particles" (described by φ) may be revealed through the intersection between the two geometrical objects $\mathcal{V}$ and $\mathcal{G}$, from which waves u_k can be emitted.

The amplification mechanism that may arise after summing the u_k's can be viewed as a *resonance*. But now, the waves u_k are no more sure to overlap. In contrast to the toy model, since $\partial_x \varphi \not\equiv 0$ and $p' \not\equiv 0$, the waves u_k do propagate in $\mathbb{R}^2$. They propagate in different directions and with various group velocities. They can mix before reaching the long times $t \sim \varepsilon^{-1}$.

FACT 3. – In the PDE framework of Equation (1.28), the accumulation of the emitted oscillating waves u_k can produce during long times $T \sim 1$ both constructive and destructive interferences.

1.3. Statement of main results

The analysis of the creation, the propagation, the linear superposition, and the nonlinear interaction of the u_k's is a manner to approach some kind of *turbulence*. We start with situations where the linear aspects are predominant. A standard Picard scheme can be used to approximate the nonlinear Equation (1.20). The corresponding first two iterates yield the Cauchy problems (1.28a) and (1.28b).

THEOREM 1.3 (Situations where the linear asymptotic behavior is predominant). – *Select a source term* F_L *as indicated in* (1.23) *with a phase* φ *depending on* γ *according to* (1.26). *Take profiles* a_m *satisfying both* (1.24) *and* (1.25). *Look at the Equation* (1.20) *with a symbol* p *subject to both* (1.21) *and* (1.22). *Introduce the profiles* $\mathcal{U}^{(j)}$, *with* $j \in \{0,1\}$, *which are issued from* (1.29) *after solving* (1.28). *Fix some* $\mathcal{T} > 0$.

The aim here is to describe the asymptotic behavior of the $\mathcal{U}^{(j)}$ *when* ε *goes to zero. Below, in* (1), *we first examine what happens in the linear case, when* $F_{NL} \equiv 0$. *Then, in* (2), *we identify nonlinearities* $F_{NL} \not\equiv 0$ *whose introduction has no impact at leading order.*

1. Linear case ($F_{NL} \equiv 0$). *Concerning the profile* $\mathcal{U}^{(0)} \equiv \mathcal{U}_{\mathrm{lin}}$, *we can produce the following distinct asymptotic behaviors when* ε *goes to zero.*
   — Constructive interferences. *For all* $j \in \mathbb{Z}$ *and* $T \in [\mathcal{T}, 2\mathcal{T}]$,

$$\mathcal{U}_{\mathrm{lin}}(T, 2j) = \mathcal{O}(1) = A_\varepsilon^2 \int_0^{+\infty} e^{-i\frac{\ell}{6}\left(\frac{1}{s}-\frac{T}{s^2}\right)} \underline{a}(s,0,0)\, ds + o(1), \tag{1.33}$$

where $A_\varepsilon^2 = \sqrt{\dfrac{2}{\pi\gamma}} e^{-i\frac{\gamma}{\varepsilon}} \cos\left(\dfrac{\gamma}{\varepsilon} - \dfrac{\pi}{4}\right)$ *is as in* (1.14).

   — Destructive interferences. *By contrast, for all* $z \in \mathbb{R} \setminus 2\mathbb{Z}$ *and for all* $T \in [\mathcal{T}, 2\mathcal{T}]$, *we find that*

$$|\mathcal{U}_{\mathrm{lin}}(T, z)| = o(1). \tag{1.34}$$

2. Nonlinear case ($F_{NL} \not\equiv 0$). *Adjust the nonlinearity* F_{NL} *as in* (1.27), *with real parameters* ν, j_1, j_2 ω *and* ι. *Assume that either* $\nu + j_1 + j_2 > 2$, *or* $\nu + j_1 + j_2 = 2$ *with* $\omega + j_1 - j_2 \neq 1$. *Fix some* $\iota \in [0,1]$. *In the case* $\nu + j_1 + j_2 - 2 = \omega + j_1 - j_2 = 0$, *set* $\iota = 1$. *Then the nonlinearity plays no role at leading order in the sense that*

$$\forall (T, z) \in [0, 2\mathcal{T}] \times \mathbb{R}, \qquad \mathcal{U}^{(1)}(T, z) = \mathcal{U}_{\mathrm{lin}}(T, z) + o(1). \tag{1.35}$$

Interpreted in the setting of SMP, Theorem 1.3 shows, as forecast in [**8**], that small plasma waves (the u_k's) driven by microscopic instabilities can accumulate over long times to furnish nontrivial effects. In turn, this phenomenon participate in some anomalous transport [**7**] and can trigger instabilities which may act as obstructions to the confinement of magnetized plasmas [**11**]. Applied in the context of NMR, our result investigates the processes whereby human tissues could be heated during magnetic resonance imaging [**21**].

It is worth noting that the turbulent aspects which are revealed by Theorem 1.3 are inherently linked to spatial heterogeneity. They are caused by the impact of the inhomogeneous source term F_L, which involves special oscillating wave front sets. Both in SMP and NMR, the input of energy is due to a strong external magnetic field $\mathbf{B}$, whose directions vary with the spatial positions, see Chapter 2.

Theorem 1.3 indicates that Facts 1, 2 and 3 indeed prevail. We still have two notions of criticality as far as nonlinear effects are concerned: the size of the nonlinearity (through the choice of $\nu + j_1 + j_2$) and the nature of oscillations (involving the gauge parameter $\mathfrak{g} = \omega + j_1 - j_2$).

The case $\nu + j_1 + j_2 > 2$ corresponds to a nonlinearity whose amplitude is too weak to have effects at leading order, regardless of the gauge. The case $\nu + j_1 + j_2 = 2$ corresponds to a nonlinearity with a critical size, for which we have to further investigate the content of the oscillations. For $\mathfrak{g} \neq 1$, that is for $\omega + j_1 - j_2 \neq 1$, the oscillations in the nonlinear term are not *resonant*. They prevent the nonlinearity from having a leading order contribution. This is why we have (1.35).

In practice, the expression (1.33) is built as a sum of wave packets, which may be viewed as corresponding to the terms u_k of (1.12). But now, the wave packets accumulate only at special positions which, in the space variable x, are located on a moving lattice of size ε. The complete statement is Proposition 4.16, which takes into account the general choices of p and F_L introduced in Chapter 2.

By contrast, at all other positions, as indicated in (1.34), the wave packets u_k compensate to furnish asymptotic disappearance. This is due to mixing properties induced by the variations of the phase ($\partial_x \varphi \not\equiv 0$) and dispersive effects ($p' \not\equiv 0$), mixing properties which are recorded in the arithmetic properties of a phase shift. This is a feature of the PDE (1.20), which is completely absent from the ODE (1.3). The full statement can be found in Proposition 4.18.

Compare (1.16) and (1.33). The characteristic function $1_{[0,T]}(s)$ of (1.16) plays the role of $\underline{a}(s,0,0)$ inside (1.33). Observe however that the Formula (1.33) differs from (1.16), due to the factor $\exp\big(-i\frac{\ell}{6}(\frac{1}{s} - \frac{T}{s^2})\big)$ in front of $\underline{a}$. This additional factor is induced by the rate of convergence of $p''(\xi)$ towards 1, which appears at the end of line (1.21). It is absent when $p \equiv 1$. In comparison to (1.16), due to the presence of an oscillating factor, it can reduce the amplification phenomenon which is revealed by (1.33). It reflects some microlocal effect, which is encoded in the behavior of p, on the asymptotic behavior of the solution $\mathcal{U}_{\text{lin}}$.

Remark that the constructive interferences (1.33) would be very difficult to detect in Lebesgue norms other than L^∞, like L^2. This is because the asymptotic profile of $\mathcal{U}_{\text{lin}}$ is nontrivial only on a set of Lebesgue measure zero (the lattice $\mathbb{Z}$). To some extent, we can say that the underlying mechanisms rely on the recombination of small scales (rapid oscillations) into larger scales, which produces (asymptotically) a very weak solution.

As already explained, the linear part (1) of Theorem 1.3 is a direct consequence of Propositions 4.16 and 4.18. The proof relies basically on classical stationary and non-stationary phase arguments to precisely describe the infinite number ($k \in \mathbb{N}$)

of emitted signals u_k. But the linear superposition of the u_k is a quite complicated mechanism. This requires to sort between dispersive and almost stationary waves, and this means to carefully examine the phase compensation phenomena that occur in the summation process. The integral inside (1.33) appears ultimately as the limit of a Riemann sum indexed by k.

The comparison between the linear solution $\mathcal{U}^{(0)} \equiv \mathcal{U}_{\rm lin}$ and the expression $\mathcal{U}^{(1)}$ is a nontrivial test to measure whether or not nonlinear effects can alter the solution at leading order. Subparagraph (2) of Theorem 1.3 deals with situations where this effect is negligible, see (1.35).

The content of (1.35) is proved in Chapter 5.2. According to the choice of $\mathfrak{g}$ or $\iota \in [0,1]$, the size of the $o(1)$ inside (1.35) may be improved, see Propositions 5.18, 5.19 and 5.20. In view of Theorem 1.3, nonlinear phenomena can be expected only under critical nonlinearities ($\nu + j_1 + j_2 = 2$) and resonant oscillations ($\mathfrak{g} = 1$).

General nonlinear source terms will be investigated in Sections 5.1 and 5.2. But, because it is simpler and already quite illustrative, in Section 5.3, we only examine the case of u^2. Other quadratic nonlinearities may be more difficult to resolve. Retain also that, higher-order nonlinearities, like the cubic choice $|u|^2 u$, appear to be not directly manageable through our approach, see Remark 5.26.

Recall that F_L has been defined at the level of (1.23). The implementation of u^2 corresponds at the level of (1.27) to the selection of $\lambda = 1$ and $(\nu, j_1, j_2) = (0,2,0)$, so that $\omega = -1$ (since we want to impose $\mathfrak{g} = 1$). Thus, we consider the solution $u^{(0)} = u_{\rm lin}$ to (1.28a), as well as the solution $u^{(1)}$ to $u^{(1)}|_{t=0} = 0$ together with

$$\partial_t u^{(1)} - \frac{i}{\varepsilon} p(-i\varepsilon\partial_x) u^{(1)} = F_L + \chi\Big(3 - 2\frac{\varepsilon t}{\mathcal{T}}\Big) \chi\left(\frac{x}{r\varepsilon^\iota}\right) e^{-it/\varepsilon} \big(u^{(0)}\big)^2. \tag{1.36}$$

Theorem 1.4 (Nontrivial nonlinear effects in the presence of resonances). – *The general context is as in Theorem 1.3. We fix $\nu = 1$, $j_1 = 2$, $j_2 = 0$ and $\omega = -1$ to deal with the quadratic source term u^2 of* (1.36). *It follows that the gauge parameter $\mathfrak{g} = \omega + j_1 - j_2 = 1$ is resonant. Select some $\iota \in]\iota_-, 1[$ with $\iota_- := (13 - \sqrt{89})/8$. Then, for all time $T \in [\mathcal{T}, 2\mathcal{T}]$ and for all position $z \in \mathbb{R}$, the expressions $\mathcal{U}^{(0)}(T,\cdot)$ and $\mathcal{U}^{(1)}(T,\cdot)$ which are issued from* (1.29) *after solving* (1.28a) *and* (1.36) *have the following asymptotic behaviors when ε goes to zero.*

— Constructive interferences. *When $z = 2j$ for some $j \in \mathbb{Z}$, the nonlinear interactions have some effect at leading order. As a matter of fact, we find*

$$\begin{aligned} \mathcal{W}^{(1)}(T,2j) &:= \mathcal{U}^{(1)}(T,2j) - \mathcal{U}^{(0)}(T,2j) \\ &= o(1) + A_\varepsilon^4 \int_0^T \chi\big(3 - 2\frac{s}{\mathcal{T}}\big) \\ &\quad \times \left(\int_0^{+\infty} \int_0^{+\infty} e^{-i\frac{\ell}{6}\frac{T-s}{(\sigma_1+\sigma_2)^2}} b(\sigma_1,s) b(\sigma_2,s) d\sigma_1 d\sigma_2 \right) ds, \end{aligned} \tag{1.37}$$

where A_ε^2 is as in (1.14) *and $b(\sigma,s) := e^{-i\frac{\ell}{6}\left(\frac{1}{\sigma} - \frac{s}{\sigma^2}\right)} \underline{a}(\sigma,0,0)$.*