Mémoires
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Numéro 175
Nouvelle série

ON THE pro-p IWAHORI HECKE Ext-ALGEBRA OF $\mathrm{SL}_2(\mathbb{Q}_p)$

R. OLLIVIER & P. SCHNEIDER

2 0 2 2

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 38 € ($ 57)
Abonnement électronique : 113 € ($ 170)
Abonnement avec supplément papier : 167 €, hors Europe : 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat

Mémoires de la SMF
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
`memoires@smf.emath.fr` • `http://smf.emath.fr/`

ISSN papier 0249-633-X; électronique : 2275-3230

ISBN 978-2-85629-944-9

`doi:10.24033/msmf.483`

Directeur de la publication : Fabien DURAND

MÉMOIRES DE LA SMF 175

ON THE PRO-p IWAHORI HECKE EXT-ALGEBRA OF $\mathrm{SL}_2(\mathbb{Q}_p)$

Rachel Ollivier

Peter Schneider

Société Mathématique de France 2022

R. Ollivier

Mathematics Department, University of British Columbia,
1984 Mathematics Road, Vancouver BC V6T 1Z2, Canada.

Courriel : `ollivier@math.ubc.ca`

P. Schneider

Mathematisches Institut, Westfälische Wilhelms-Universität Münster,
Einsteinstr. 62, 48149 Münster, Germany.

Courriel : `pschnei@uni-muenster.de`

Soumis le 4 mai 2021, révisé le 14 août 2022, accepté le 5 octobre 2022.

2000 *Mathematics Subject Classification.* – 20C08, 22E50, 16E30, 20J06, 11F85.

***Key words and phrases.* –** Cohomology of the pro-p Iwahori subgroup, Ext-algebra, pro-p Iwahori Hecke algebra, representations of p-adic SL_2, supersingularity.

***Mots clefs.* –** Cohomologie du pro-p Iwahori, Ext-algèbre, algèbre de Hecke du pro-p Iwahori, représentations du groupe p-adique SL_2, supersingularité.

ON THE PRO-p IWAHORI HECKE EXT-ALGEBRA OF $\mathrm{SL}_2(\mathbb{Q}_p)$

Rachel Ollivier, Peter Schneider

Abstract. – Let $G = \mathrm{SL}_2(\mathfrak{F})$ where $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$. We suppose that the pro-p Iwahori subgroup I of G is a Poincaré group of dimension d. Let k be a field containing the residue field of $\mathfrak{F}$.

In this volume, we study the graded Ext-algebra $E^* = \mathrm{Ext}^*_{\mathrm{Mod}(G)}(k[G/I], k[G/I])$. Its degree zero piece E^0 is the usual pro-p Iwahori-Hecke k-algebra H.

We study E^d as an H-bimodule and deduce that for an irreducible admissible smooth k-representation V of G, we have $H^d(I, V) = 0$ unless V is the trivial representation.

When $\mathfrak{F} = \mathbb{Q}_p$ with $p \geq 5$, we have $d = 3$. In that case we describe E^* as an H-bimodule and give the structure as an algebra of the centralizer in E^* of the center of H. We deduce results on the values of the functor $H^*(I, \_)$ which attaches to a (finite length) smooth k-representation V of G its cohomology with respect to I. We prove that $H^*(I, V)$ is always finite dimensional. Furthermore, if V is irreducible, then V is supersingular if and only if $H^*(I, V)$ is a supersingular H-module.

Résumé **(Sur la Ext-algèbre de Hecke du pro-p Iwahori de $\mathrm{SL}_2(\mathbb{Q}_p)$)**

Soit $G = \mathrm{SL}_2(\mathfrak{F})$ où $\mathfrak{F}$ est une extension finite $\mathbb{Q}_p$. On suppose que le sous-groupe d'Iwahori I de G est un groupe de Poincaré de dimension d. Soit k un corps contenant le corps résiduel de $\mathfrak{F}$.

Dans ce texte, nous étudions la Ext-algèbre graduée $E^* = \mathrm{Ext}^*_{\mathrm{Mod}(G)}(k[G/I], k[G/I])$. Sa composante de degré zero est la k-algèbre de Hecke du pro-p Iwahori H.

Nous étudions le H-bimodule E^d et déduisons que, étant donnée une k-représentation irréductible admissible lisse V de G, on a $H^d(I, V) = 0$ à moins que V ne soit la représentation triviale.

Lorsque $\mathfrak{F} = \mathbb{Q}_p$ avec $p \geq 5$, on a $d = 3$. Dans ce cas, nous décrivons le H-bimodule E^* et la structure d'algèbre du centralisateur dans E^* du centre de H. Nous en déduisons des résultats quant aux valeurs du foncteur qui attache à une k-représentation lisse (de longueur finie) V de G l'espace de I-cohomologie $H^*(I, V)$. Nous montrons que $H^*(I, V)$ est toujours de dimension finie. De plus, si V est irréductible, alors V est supersingulière si et seulement si $H^*(I, V)$ est un module supersingulier.

CONTENTS

CHAPTER 1

INTRODUCTION

Let $\mathfrak{F}$ be a locally compact nonarchimedean field with residue characteristic p, and let G be the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$. We suppose that $\mathbf{G}$ is $\mathfrak{F}$-split.

Let k be a field of characteristic p and let $\mathrm{Mod}(G)$ denote the category of all smooth representations of G in k-vector spaces. For a general $\mathbf{G}$ and $\mathfrak{F}$ this category is still poorly understood. One way of approaching it consists in considering the Hecke algebra H of the pro-p Iwahori subgroup $I \subset G$. In this case the natural left exact functor

$$\begin{aligned} \mathfrak{h} : \mathrm{Mod}(G) &\longrightarrow \mathrm{Mod}(H) \\ V &\longmapsto V^I = \mathrm{Hom}_{k[G]}(\mathbf{X}, V) \end{aligned}$$

sends a nonzero representation onto a nonzero module. Its left adjoint is

$$\begin{aligned} \mathfrak{t} : \mathrm{Mod}(H) &\longrightarrow \mathrm{Mod}^I(G) \subseteq \mathrm{Mod}(G) \\ M &\longmapsto \mathbf{X} \otimes_H M. \end{aligned}$$

Here $\mathbf{X}$ denotes the space of k-valued functions with compact support on G/I with the natural left action of G. The functor $\mathfrak{t}$ has values in the category $\mathrm{Mod}^I(G)$ of all smooth k-representations of G generated by their I-fixed vectors. This category, which a priori has no reason to be an abelian subcategory of $\mathrm{Mod}(G)$, contains all irreducible representations. But in general $\mathfrak{t}$ is not an equivalences of categories and little is known about $\mathrm{Mod}^I(G)$ and $\mathrm{Mod}(G)$ unless $G = \mathrm{GL}_2(\mathbb{Q}_p)$ or $G = \mathrm{SL}_2(\mathbb{Q}_p)$ (**[6]**, **[11]**, **[13]**, **[16]**).

The functor $\mathfrak{h}$, although left exact, is not right exact since p divides the pro-order of I. It is therefore natural to consider the derived functor. In **[17]** the following result is shown: When $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$ and I is a torsion free pro-p group, there exists a derived version of the functors $\mathfrak{h}$ and $\mathfrak{t}$ providing an equivalence between the derived category $D(G)$ of smooth representations of G in k-vector spaces and the derived category of differential graded modules over a certain differential graded pro-p Iwahori-Hecke algebra $H^\bullet$.

The article **[14]** opened up the study of the Hecke differential graded algebra $H^\bullet$ by giving the first results on its cohomology algebra $E^* := \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$. This is the pro-$p$ Iwahori Hecke Ext-algebra we refer to in the title of the current article. We suppose in this introduction that I is a torsion free p-adic Lie group which forces $\mathfrak{F}$ to be a finite extension of $\mathbb{Q}_p$. We denote by d the dimension of I as a Poincaré group. The Ext algebra E^* is supported in degrees 0 to d.

When $\mathbf{G}$ is almost simple and simply connected, the ideal $\mathfrak{J}H$ which controls the supersingularity (see §2.1) has finite codimension in H. We show that we have an isomorphism of H-bimodules

$$\mathrm{Ext}^d_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X}) \cong \chi_{\mathrm{triv}} \oplus \mathrm{Inj}((H/\mathfrak{J}H)^\vee), \tag{1}$$

where χ_{triv} is the trivial character of H and $\mathrm{Inj}((H/\mathfrak{J}H)^\vee)$ is an injective envelope of the dual module $(H/\mathfrak{J}H)^\vee$. When $\mathbf{G} = \mathrm{SL}_2$, the center of H contains a polynomial algebra $k[\zeta]$ and $\mathfrak{J}H = \zeta H$. The large injective module inside of $\mathrm{Ext}^d_{\mathrm{Mod}(G)}(\mathbf{X}, \mathbf{X})$ is ξ-divisible for any $\xi \in H$ which is a non-zero-divisor. This, together with the decomposition (1), allows us to prove (Cor. 2.19) that given Q a nonzero polynomial in $k[X]$, we have $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) = 0$ unless $Q(1) = 0$ in which case $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) \cong \chi_{\mathrm{triv}}$. But we remark that every irreducible admissible representation of $\mathrm{SL}_2(\mathfrak{F})$ is a quotient $\mathbf{X}/\mathbf{X}Q(\zeta)$ for some Q as above and we prove:

PROPOSITION (Proposition 2.20). – *We have $H^d(I, V) = 0$ for any irreducible admissible representation of $\mathrm{SL}_2(\mathfrak{F})$ except when V is the trivial representation in which case $H^d(I, k_{\mathrm{triv}}) \cong \chi_{\mathrm{triv}}$ as an H-bimodule.*

In Sections 3 and 4, we move on to the study of E^1 and E^{d-1} respectively. Here we fully use the Frobenius reciprocity recalled in §2.2 which allows to identify E^i with $H^i(I, \mathbf{X})$. We decompose the latter, via the Shapiro isomorphism, as a direct sum

$$\bigoplus_{w \in \widetilde{W}} H^i(I_w, k)$$

where w ranges over $\widetilde{W}$ (defined at the beginning of Section 2, see also §2.4.1) and $I_w = I \cap wIw^{-1}$. We explain in §3.2 that we see elements of $H^1(I_w, k)$ as triples. This is valid for $G = \mathrm{SL}_2(\mathfrak{F})$ with no restriction on $\mathfrak{F}$ and stems from the computation of the Frattini quotient of I_w. When I is a Poincaré group of dimension d, we use the duality between E^1 and E^{d-1} (§14) to also express elements of $H^{d-1}(I_w, k)$ as triples in §4.1. When $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$, Remark §3.2 points out that the triples of $H^1(I_w, k)$ are simply the elements in

$$\mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k) \times \mathrm{Hom}((1 + p\mathbb{Z}_p)/(1 + p^2\mathbb{Z}_p), k) \times \mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k)$$

hence by duality the triples of $H^2(I_w, k)$ are the elements in

$$\mathbb{Z}_p/p\mathbb{Z}_p \otimes_{\mathbb{F}_p} k \times ((1 + p\mathbb{Z}_p)/(1 + p^2\mathbb{Z}_p)) \otimes_{\mathbb{F}_p} k \times \mathbb{Z}_p/p\mathbb{Z}_p \otimes_{\mathbb{F}_p} k.$$

In this context, the full left action of H on the triples of E^1 and of E^2 can be found in §3.6 and §4.3 (the proof of the most technical formulas is postponed to the appendix).

The right action of H on the triples can be deduced using the anti-involution $\mathcal{J}$ of E^* (see §2.2.3 and Lemmas 3.7 and 4.1). We are especially interested in the left and right action of the central element $\zeta \in H$ (which is fixed by $\mathcal{J}$).

In Section 5 we study the $k[\zeta]$-torsion on the left in certain graded pieces of E^* when $G = \mathrm{SL}_2(\mathfrak{F})$, with various restrictive conditions on $\mathfrak{F}$ depending on the graded piece in question. Only for the computation of the $k[\zeta]$-torsion in E^2 do we use the explicit formulas for the action of ζ hence we have to restrict ourselves to $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$.

Contemplating the formulas for the action of ζ on E^1 and E^2 (still when $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$) emphasizes the role of the operators

$$f := \zeta \cdot \mathrm{id}_{E^*} \cdot \zeta - \mathrm{id}_{E^*} \quad \text{and} \quad g := \zeta \cdot \mathrm{id}_{E^*} - \mathrm{id}_{E^*} \cdot \zeta$$

as introduced in §6.1. The kernel of f is a $k[\zeta^{\pm 1}]$-bimodule. Describing its structure as an H-bimodule requires the technical Paragraph 3.7.3.2 (then see Propositions 6.8, 6.19 and Lemma 6.2). On the other hand, as the centralizer in E^* of ζ, the kernel of g is naturally a subalgebra of E^*. We describe this kernel precisely in §6.2.1 and §6.3.1 (and Lemma 6.2) and conclude in Proposition 8.1 that it actually coincides with the centralizer $\mathcal{C}_{E^*}(Z)$ of the whole center $Z := Z(H)$ of H in E^*. The product in this natural subalgebra of E^* is explicitly given in Section 8. (Note that the center of H is no longer central in E^*).

Proposition 6.13 says that E^2 is, as an H-bimodule, isomorphic to the direct sum of the kernels of the operators f and g (restricted to E^2) and Proposition 6.10 says that it is also (essentially) the case for E^1. This allows us to completely determine the structure of E^* as a left and right $k[\zeta]$-module (Proposition 7.2) and to establish results such as Proposition 7.6 where we study the $k[\zeta]$-torsion on the left in spaces of the form $H^*(I, \mathbf{X}/\mathbf{X}Q(\zeta))$ for $Q \in k[X]$. This in particular leads to the following theorem:

THEOREM (Theorem 7.11). – *Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. For any representation of finite length in* $\mathrm{Mod}(G)$ *we have:*

i. *The k-vector space $H^*(I, V)$ is finite dimensional;*
ii. *if V is generated by its subspace of I-fixed vectors V^I and $Q(\zeta)V^I = 0$ for some nonzero polynomial $Q \in k[X]$, then the left H-module $H^*(I, V)$ is $P(\zeta)$-torsion for the polynomial $P(X) := Q(X)Q(\frac{1}{X})X^{\deg(Q)}$.*

The most interesting consequence of this theorem is that, under the same hypotheses, an irreducible representation V in $\mathrm{Mod}(G)$ is supersingular if and only if the left H-module $H^*(I, V)$ is supersingular (this is Corollary 7.12 which uses the theorem in the case when $Q = X$). This strongly indicates that the notion of supersingularity for general G can be extended to objects in the derived category $D(G)$ by introducing a theory of supports via the dg algebra $H^\bullet$. We hope to return to this in another paper.

In **[13]** §3.5 we studied the representation theoretic meaning of the localization H_ζ of the Hecke algebra in the central element ζ. Despite the fact that ζ is no longer central in E^* it turns out (Remark 7.7) that $\zeta^{\mathbf{N}_0}$ is a left and right Ore set in E^*, so that the localization E_ζ^* does exist. We will show elsewhere that E_ζ^* again is a Yoneda Ext-algebra and will investigate its meaning for the nonsupersingular $\mathrm{SL}_2(\mathbb{Q}_p)$-representations.

After this paper was finished E. Bodon ([**2**]) gave in his thesis, building very much on the computational methods developed in the present paper, two further structural results in the case $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. He describes explicitly the full graded center of E^*. Even more remarkably he shows that the algebra E^* as an algebra over H is finitely presented.

In forthcoming work of the second author with K. Ardakov we develop a general theory of central spaces for a certain class of Grothendieck categories which refines the notion of the center of an abelian category. It was shown in [**1**] that the usual center of the category $\mathrm{Mod}(G)$ is very small. For example, if $\mathbf{G} = \mathrm{SL}_2$ then this center is the group ring $k[Z(G)]$ of the center of G. In contrast the central space in this case with $\mathfrak{F} = \mathbb{Q}_p$ is a projective variety over k which is a quotient of the affine variety $\mathrm{Spec}(Z)$ by a relation which is given by the annihilator ideal of the $Z \otimes_k Z$-bimodule E^*. The results of the present paper allow to compute this ideal and therefore this projective variety explicitly. Therefore we strongly believe that this bimodule and its support variety play a basic role for the computation of the central space of $\mathrm{Mod}(G)$ for general groups G.

This collaboration was partially funded by the NSERC Discovery Grant of the first author and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 390685587, Mathematics Münster: DynamicsGeometryStructure. Both authors also acknowledge the support of PIMS at UBC Vancouver.