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On Lusztig's parametrization of characters of finite groups of Lie type

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This paper has three parts. In the first part, we extend Lusztig's results of [11] about the parametrization of characters of finite reductive groups with a connected center, including [11, theorem 4.23] about multiplicities of irreducible characters in the Deligne-Lusztig characters, to the case of groups with non-connected center. We use mostly a method sketched in chapter 14 of [11] in the case of a cyclic center, based on Clifford theory and a result about the unicity of the parametrisation of characters constructed in [11] which we prove in section 6 (part II). This construction has been carried out by Lusztig in [13] but we need more information than he gets there, in order to get the results of section 5 and of part III.

In sections 1 and 2 we state the results we need from Clifford theory, from [11] and about non-connected groups. We also need a result about the commutation of Lusztig twisted induction with isogenies, whose proof is given in section 9 (part III) using Shintani descent. We then apply these results to the parametrization of characters in section 3, where we need the results of part II. Finally section 4 and section 5 describe the multiplicities of irreducible characters in Deligne-Lusztig characters using Lusztig "families", presented here from a simplified combinatorial viewpoint using the "Mellin transform".

Part II describes under which conditions Lusztig's parametrization of irreducible characters in [11] is unique; section 6 deals with families and Weyl groups, and section 7 gives the main theorem.

Part III studies Shintani descent in groups with non-connected center. We want to show how Shintani descent relates to the parametrization introduced in part I. Section 8 recalls facts about Shintani descent and "F'-twisted induction". In section 9 we prove a result about the commutation of F'-twisted induction with isogenies and deduce the analogous result for Lusztig's twisted induction. In section 10, we first extend to F-class functions the parametrization of section 5 (when the center is not connected we have to make assumptions that we cannot yet prove in all cases). Finally, we give a formula for Shintani descent of principal series characters using the Fourier transform on families (using section 5 of part I). For this last result we have to quote heavily from [7].

This paper * has been prompted by discussions with B. Srinivasan, and Shoji's papers [14] and [15] where he gets a complete description of Shintani descent $\mathrm{Sh}_{F^m/F}$ for m sufficiently divisible and for a group with connected center (Shoji himself uses results of Asai [1] which deal with the case $m=1$); this paper also has been prompted by the absence of a convenient written description dealing with groups with non-connected center.

0. Background.

In this section we recall some results from Clifford theory and the theory of F-class functions.

We denote by $\mathrm{Irr}(G)$ the set of irreducible characters of the finite group G (over an algebraically closed field of characteristic 0). We now give a general proposition which states basic (well known) results from Clifford theory. Most of these are easy consequences of Mackey formula and Frobenius reciprocity (see also [8, 2.1]).

0.1 PROPOSITION (CLIFFORD THEORY). *Let G be a normal subgroup of a finite group $\tilde G$ such that the quotient $\tilde G/G$ is abelian; let $\tilde Z$ be the center of $\tilde G$. For $\rho\in\mathrm{Irr}(\tilde G)$, we put $A(\rho)=\{\zeta\in\mathrm{Irr}(\tilde G/G\tilde Z)\mid \rho\otimes\zeta=\rho\}$ (note that $\tilde Z$ is in the kernel of any $\zeta\in\mathrm{Irr}(\tilde G/G)$ such that $\rho\otimes\zeta=\rho$). If $\mu\in\mathrm{Irr}(G)$ is a component of $\mathrm{Res}^{\tilde G}_G\rho$, we note $\tilde G(\rho)$ for the inertia group of μ in $\tilde G$ (it depends only on ρ (not on μ)). Then we have:*

(i) $\mathrm{Ker}(A(\rho))\subset\tilde G(\rho)$.

(ii) There exists $\tilde\mu\in\mathrm{Irr}(\mathrm{Ker}(A(\rho)))$, $\tilde\rho\in\mathrm{Irr}(\tilde G(\rho))$ and a positive integer e such that:

$$\mathrm{Ind}_G^{\mathrm{Ker}(A(\rho))}(\mu)=\sum_{\alpha\in\mathrm{Irr}(\mathrm{Ker}(A(\rho))/G)}\tilde\mu\otimes\alpha,\mathrm{Res}_G^{\mathrm{Ker}(A(\rho))}(\tilde\mu)=\mu,$$

$$\mathrm{Ind}_{\mathrm{Ker}(A(\rho))}^{\tilde G(\rho)}\tilde\mu=e\tilde\rho,\qquad \mathrm{Res}_{\mathrm{Ker}(A(\rho))}^{\tilde G(\rho)}\tilde\rho=e\tilde\mu,$$

$$\mathrm{Ind}_{\tilde G(\rho)}^{\tilde G}(\tilde\rho)=\rho,\qquad \mathrm{Res}_{\tilde G(\rho)}^{\tilde G}(\rho)=\sum_{x\in\tilde G/\tilde G(\rho)}{}^x\tilde\rho,$$

* part of this work was done during the authors visit at Essen university

(iii) The quotient group $\tilde{G}(\rho)/\mathrm{Ker}(A(\rho))$ has cardinality e^2 and we have

$$|A(\rho)| = \langle \mathrm{Res}_G^{\tilde{G}}(\rho), \mathrm{Res}_G^{\tilde{G}}(\rho) \rangle_G.$$

If $\tilde{G}(\rho)/\mathrm{Ker}(A(\rho))$ is cyclic, then $e = 1$.

The following result, which is proved in [7, 6.1] gives $e = 1$ in a general setting for Weyl groups:

0.2 LEMMA. *Assume that G is a Weyl group and $\tilde{G}$ is the semi-direct product of G by a group A of diagram automorphisms of G, then for any character $\rho \in \mathrm{Irr}(\tilde{G})$ we have $e = 1$.*

The proof of this lemma requires the following result (*cf.* [7, 6.2]) that we will need below in the proof of 5.5

0.3 LEMMA. *Let G be a finite group of the form $G_1 \times \ldots \times G_l$ and A be a finite group of automorphisms of G acting by permutation of the G_i. Let $\mu = \mu_1 \otimes \ldots \otimes \mu_l$ be an irreducible character of G. Let A_i be the subgroup of $\mathrm{Stab}_A(\mu)$ normalizing G_i (and so μ_i). If for each i, the character μ_i has an extension to $G_i \rtimes A_i$, then μ has an extension to $G \rtimes \mathrm{Stab}_A(\mu)$ (i.e., $e = 1$ for the character μ).*

0.4 F-CLASS FUNCTIONS. If G is a finite group and if $<F>$ is a group generated by an element F and acting on G, we denote by $\mathcal{C}(G/F)$ the space of complex valued F-class functions on G, i.e. functions φ which verify $\varphi(x.{}^F y) = \varphi(yx)$ for any x and y in G (note that the group $<F>$ can be infinite). We may identify $\mathcal{C}(G/F)$ with the space of restrictions to the set $G.F$ of class functions on the semi-direct product $G \rtimes <F>$. This space admits as a basis the set of restrictions to $G.F$ of an arbitrarily chosen extension to $G \rtimes <F>$ of each F-invariant irreducible character of G. If φ_1 and φ_2 are elements of $\mathcal{C}(G/F)$, we put

$$\langle \varphi_1, \varphi_2 \rangle_{G.F} = |G|^{-1} \sum_{x \in G.F} \overline{\varphi_1(x)} \varphi_2(x).$$

We recall that if φ_1 and φ_2 are characters of $G \rtimes <F>$ whose restrictions to G are irreducible, then

$$\langle \varphi_1, \varphi_2 \rangle_{G.F} = \begin{cases} 0, & \text{if } \mathrm{Res}_G^{G \rtimes <F>} \varphi_1 \neq \mathrm{Res}_G^{G \rtimes <F>} \varphi_2 \\ 1, & \text{if } \varphi_1 = \varphi_2 \end{cases}$$

(if φ_1 and φ_2 have equal restrictions to G, then they differ by multiplication by a linear character of $<F>$).

If H is a subgroup of G stabilized by F, we denote by $\mathrm{Res}_{H.F}^{G.F}$ the restriction of F-class functions, and we define induction of F-class function by

$$\mathrm{Ind}_{H.F}^{G.F}(f)(gF) = |H|^{-1} \sum_{\{\gamma \in G \mid {}^\gamma(gF) \in H.F\}} f({}^\gamma(gF)).$$

Induction and restriction are adjoint with respect to the above scalar product.

I

1. Disconnected groups.

In this section we extend the definition of Deligne-Lusztig characters to non connected reductive groups. We begin with a proposition which gives the relation between the Weyl group of a reductive group and that of its connected component.

1.1 PROPOSITION. *Let* $\mathbf{H}$ *be a reductive algebraic group, and let* $\mathbf{T}$ *be a maximal torus of* $\mathbf{H}$*; then we may find representatives of* $\mathbf{H}/\mathbf{H}^\circ$ *in* $N_{\mathbf{H}}(\mathbf{T})$*. We set* $W = N_{\mathbf{H}}(\mathbf{T})/\mathbf{T}$ *and* $W^\circ = N_{\mathbf{H}^\circ}(\mathbf{T})/\mathbf{T}$*. Let* $\mathbf{B}$ *be a Borel subgroup containing* $\mathbf{T}$*. We put* $A = \{w \in W \mid {}^w\Phi^+ = \Phi^+\}$ *where* Φ *is the root system of* $\mathbf{H}^\circ$ *and* $+$ *denotes the order on* Φ *corresponding to* $\mathbf{B}$*. Then we have*

(i) $W = W^\circ \rtimes A$ *and* $A \simeq \mathbf{H}/\mathbf{H}^\circ$.

(ii) If $\mathbf{H}$ *is defined over* $\mathbb{F}_q$*, with corresponding Frobenius* F*, and* F *stabilizes* $\mathbf{T}$ *and* $\mathbf{B}$ *above, then* F *stabilizes* W*,* W° *and* A*.*

In the following we consider a (not necessarily connected) reductive algebraic group $\mathbf{H}$ defined over $\mathbb{F}_q$, and denote by F the corresponding Frobenius endomorphism. We fix a pair $\mathbf{T} \subset \mathbf{B}$ of an F-stable maximal torus in $\mathbf{H}$ included in an F-stable Borel subgroup. Let $\mathbf{H}^{\circ *}$ be a group dual to $\mathbf{H}^\circ$ containing a given torus $\mathbf{T}^*$ dual to $\mathbf{T}$. Finally we fix a Frobenius endomorphism F^* dual to F. We may identify W° with $N_{\mathbf{H}^{\circ *}}(\mathbf{T}^*)/\mathbf{T}^*$ by mapping w to the dual isogeny w^*, but note that this map is an anti-isomorphism. For any $v \in W^\circ$ we choose a representative $\dot{v}^* \in N_{\mathbf{H}^{\circ *}}(\mathbf{T}^*)$ of v^* and for any representative $\dot{a} \in N_{\mathbf{H}}(\mathbf{T})$ of an element $a \in A$, we choose an isogeny $(\dot{a}F)^*$ dual to $\dot{a}F$. For $w \in W$, if w is in the coset $W^\circ a$, we write $(\dot{w}\dot{a}F)^*$ for $(\dot{a}F)^*\dot{w}^*$.