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SOME NEW BLOCK INVARIANTS COMING FROM COHOMOLOGY

BY

Christine BESSENRODT

1 Introduction

In the usual setup of the representation theory of finite groups we are given a finite group G and a ring A of coefficients, and we want to study the modules over the ring $\mathcal{A} = AG$. Typical coefficient rings are the ring $\mathbb{Z}$, the p-adic numbers $\mathbb{Z}_p$, or fields. For many properties of these modules, we can 'forget' the group G and just need to know the algebra $\mathcal{A}$. Now suppose that p is a prime dividing the order of G, and let A be a complete discrete valuation ring with residue field of characteristic p or a field of characteristic p. There are some very fruitful invariants in integral and modular representation theory which are defined with explicit reference to the given group G. The most prominent among these are the vertex of an indecomposable AG-module and the defect group of a p-block, or the kernels of modules and blocks.

Now it is natural to ask:

(1) What informations on G can we read off from $\mathcal{A}$?

(2) What happens to the invariants mentioned above, if we choose another group basis in $\mathcal{A}$, i.e. a subgroup $H \leq U(\mathcal{A})$ such that $\mathcal{A} = AH$ and $|H| = |G|$?

In particular, question (1) includes the classical isomorphism problem which was formulated by G. Higman in 1940 and later also posed by Brauer [7]:

$$\text{Does } \mathbb{Z}G \simeq \mathbb{Z}H \text{ imply } G \simeq H \text{ ?}$$

This problem has stimulated a lot of research, and the last few years have seen quite some progress, in particular in the work of Roggenkamp and Scott [22]. They have obtained positive answers for some classes of groups also to the much stronger Zassenhaus conjecture, which asks whether another (normalised) group basis for $\mathbb{Z}G$ must even be conjugate to G by a unit in $\mathbb{Q}G$. For more details and the history of the isomorphism problem the reader is referred to the articles by Roggenkamp and Scott, the books by Passman [20] and Sehgal [25], and the survey article by Sandling [23].

Roggenkamp and Scott have also dealt with other integral coefficient rings, such as the p-adic numbers $\mathbb{Z}_p$. For these, too, they could prove the Zassenhaus conjecture for nilpotent groups. For $A = \mathbb{Z}_p$ and G a p-group, Weiss [26] succeeded in proving the strong theorem that any finite subgroup of $V(\mathbb{Z}_pG)$, the augmentation 1 units in $\mathbb{Z}_pG$, is conjugate in $V(\mathbb{Z}_pG)$ to a subgroup of G. For $A = F$ a field of characteristic p, it is still an open question whether the group algebra of a p-group determines the group G. The earliest result to this question goes back to Deskins [13], who proved that an abelian p-group is determined by its modular group algebra. It is also known that the answer is positive for small p-groups and for various special classes of p-groups. The proofs are usually rather computational, and it seems hard to transfer them from the case of p-groups to general groups.

So for these coefficient rings there are rather few results to question (1) for general finite groups. On the other hand, by using the classification of the finite simple groups, Kimmerle-Lyons-Sandling [17] showed that $\mathbb{Z}G$ determines the composition factors of G. They also proved that $\mathbb{Z}G$ determines whether the Sylow subgroups of G are abelian, hamiltonian or of certain other types, and in these cases they can obtain the structure of these groups [16].

For a coefficient ring like $\mathbb{Z}_p$ or a field of characteristic p, there is at least some hope that the group ring AG determines the structure of a Sylow p-subgroup. Motivated by the recent successes, Scott asked the following more general question, which is of type (2) (see [24]):

> Given a p-block B of $\mathbb{Z}_pG$, are its defect groups determined up to conjugation and 'suitable' normalisation, independently of the group G ?

Also, Alperin pointed out that it is even open whether the isomorphism type of the defect groups is determined by B.

In our investigation we will focus mainly on the modular group algebra FG; of course, this also implies results for the integral situation.

In the following sections we present a contribution to the question posed by Scott and Alperin. Our leading idea will be that the problem of determining the isomorphism type of a defect group of a block falls into two parts: first one would like to obtain the defect group algebra from the block algebra, and then one needs a positive answer to the isomorphism problem for p-groups (as mentioned above, this is true for $\mathbb{Z}_p$, but open for fields of characteristic p). In fact, we will be more modest, and we will just try to compute certain new invariants of the defect group algebra from the block algebra. It turns out that for many types of p-groups these invariants are the same for the defect group algebra and the block algebra, and in the abelian case they even suffice to determine the isomorphism type of the defect group.

Here are a few more details on the course of our investigations. As computations inside the group algebra can usually not easily be translated to the block situation, we introduce a new tool coming from cohomology theory in the second section. For this, we use the complexity of a module, which is a measure for the growth of the dimensions of the projective modules in a minimal projective resolution for the module. This invariant was introduced by Alperin in 1977, and it has attracted much attention since Alperin and Evens [1] have proved their celebrated theorem that the complexity of a module can be determined on the elementary abelian p-subgroups. If $A = F$ is an algebraically closed field, it can also be described as the dimension of a certain variety associated with the module, which was defined by Carlson [11], who also proved many important properties of this variety.

For our purposes, we define a sequence of invariants for a p-block B (or more generally for a union of p-blocks) by looking at the dimensions of modules with a certain complexity belonging to the block B. A few properties of the defect group can easily be read off this sequence, like its order and its rank. The invariants for the whole group algebra are the same as those for the group algebra over a Sylow p-subgroup. We then show that for a defect group for which the invariants already come from trivial source modules, the invariants of the block are the same as those of the defect group algebra. Based on some results of Carlson, one can prove that for the group algebra of an abelian p-group our invariants determine the isomorphism type of the p-group, and we see that they come from trivial source modules. Thus, in particular, the structure of an abelian defect group can be deduced from the invariants for the block (see [5]), but also some other types of p-groups can be handled with this method. Unfortunately, our invariants can not decide whether the defect group is abelian, we have to assume this in advance. In fact, note that so far it is not even known if the whole modular group algebra determines whether the Sylow p-subgroups are abelian. In the last section we calculate the sequence of invariants for various p-groups.

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Let us fix some notation for the following. By G we will always denote a finite group, and by F a field of characteristic $p > 0$. Furthermore, R will always be a complete discrete valuation ring of characteristic 0 with residue field of characteristic $p > 0$, which we will then also denote by F. We assume that the quotient field of R is sufficiently large relative to G, so that it is a splitting field for G and its subgroups. The ring A will be one of the rings R or F, and an AG-module is always supposed to be finitely generated and free over A. For an AG-module M we denote by $c_G(M)$ the complexity of M (see e.g. [3]). For $n \in \mathbb{N}$ we write n_p for the highest p-power dividing n. Other standard notations and terminology may be found in the books by Benson [3] and Feit [14].

2 Some new invariants for group algebras and blocks

In this section we want to introduce some new invariants for blocks and group algebras, which are derived from looking at modules of a certain complexity; we refer the reader to Benson [3] and the papers by Alperin-Evens [1], Avrunin-Scott [2] and Carlson [11] for the properties of the complexity and the variety of a module. For the isomorphism problem we want to exploit the relationship between the complexity and the rank of an AG-module.

Now let us come to the precise definition of our invariants.