Astérisque

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Astérisque, tome 181-182 (1990), p. 173-189

<http://www.numdam.org/item?id=AST_1990__181-182__173_0>

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Table of contents

1. Introduction

Let G be a finite group, p a prime and B a p-block of G. We are interested in two different approaches to representation theory: functional and algebra theoretic.

The first deals with matrix representations and functions on groups; it includes questions about $k(B)$, the number of irreducible complex characters of B, and about $l(B)$, the number of irreducible Brauer characters of B. The second approach views the block B as a finite dimensional algebra. It is concerned with the module category of B including homological properties such as projective resolutions and the Auslander-Reiten quiver (see chapter 2).

A number of years ago, Brauer and Olsson studied 2-blocks B whose defect groups are dihedral or semidihedral or (generalized) quaternion from the functional point of view. They were interested in determining $k(B)$, $l(B)$ and to obtain information concerning the (generalized) decomposition numbers of B [4, 13].

These are also precisely the blocks which are of tame representation type [2]; and they have recently been studied from the algebra point of view. By using Auslander-Reiten theory it has been possible to classify these blocks, as algebras, by generators and relations, up to Morita equivalence (and some scalars in socle relations). In particular, this gives the Cartan matrices for all these blocks, and it allows one to calculate the decomposition numbers, hence to extend the classical results [7 to 11] of the functional approach.

The original arguments used some of the work by Brauer and Olsson from [4, 13]; however this is not necessary. The aim of this paper is to show how results on the algebras and a few general principles determine l(B), k(B) and the decomposition matrices.

We will now introduce the algebras which were studied. Let K be an algebraically closed field and Λ a finite-dimensional K-algebra.

(1.1) We say that Λ is of "dihedral" or "semidihedral" or "quaternion" type if it satisfies the following conditions:

(a) Λ is tame, symmetric and indecomposable.

(b) The Cartan matrix of Λ is non-singular.

(c) The stable Auslander-Reiten quiver of Λ has the following components:

	dihedral type	semidihedral type	quaternion type
tubes	1-tubes and	1-, 2-tubes and	1-, 2-tubes
	$\leq$ two 3-tubes	$\leq$ one 3-tube	-
others	$\mathbb{Z}A_\infty^\infty$	$\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}D_\infty$	-
	(or $\mathbb{Z}\tilde{A}_{12}$ or $\mathbb{Z}\tilde{A}_5$)		

The class of these algebras contains all dihedral, semidihedral and quaternion blocks [7, 11]. To prove this, one needs, apart from general principles, the algebra structure of some local blocks.

A main step in the classification of these algebras consists in bounding the number of simple modules. We have

THEOREM *Let Λ be an algebra of dihedral or semidihedral or quaternion type. Then Λ has at most three simple modules.*

COROLLARY [Brauer, Olsson] *Suppose that B is a dihedral or semidihedral or quaternion block. Then $l(B) \leq 3$.*

In chapter 2, we shall give a proof of the Theorem for the dihedral case. The quaternion case has been done in [9]; and the semidihedral case which is somewhat longer will appear in [11].

The work to determine the algebras is more general and is independent of groups. The results may be found in [7, 8, 10, 11]; see also [6].

In the third chapter, we will calculate Cartan matrices and decomposition matrices D for all tame blocks having two simple modules.

The information we need to do this from the classification of algebras is summarized at the beginning. It is in fact convenient to study all tame blocks simultaneously, since the same Cartan matrices appear, and since the dimension of the centre of the algebra depends only on the Cartan matrix C. Given C and k(B), to calculate D one needs (almost) nothing.

We remark that the results on the heights of characters by Brauer and Olsson follow easily from the decomposition numbers, using the general fact that any block must contain an ordinary and also a Brauer character of height zero.

The dihedral case has not been published; the results for the other types are contained in [8, 10]. However, the proofs there use results by Olsson [13].

With the same techniques, one can deal with the case l(B) = 3; this is not more difficult, though it takes longer due to the number of algebras. This will appear in [11]. Blocks with l(B) = 1 create no problem.

We write $\delta(B)$ for a defect group of the block B. If M is a module then soc M is the largest semisimple submodule of M, and top M denotes its largest semisimple factor module. Any other notation should be standard. Concerning basic facts on blocks, algebras and representations we refer to [1, 12].