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ODILE GAROTTA

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INTRODUCTION

Let G be a finite group, k an algebraically closed field whose characteristic p divides the order of G, and denote by A a symmetric interior G-algebra (that is a symmetric k-algebra A together with a homomorphism $\phi\colon G \to A^{\times}$). In [**2**], we introduce the notion of *Auslander-Reiten system* of G over A (cf. §1), as a "generalization" in terms of idempotents of the usual notion of almost split sequence of kG-modules, which then corresponds to the case where A is the algebra of k-endomorphisms of a kG-module. Furthermore we show in [**2**] that every primitive idempotent i of A^G such that $i \notin A_1^G$ is the right extremity of a unique Auslander-Reiten system, up to embedding of A into a symmetric interior G-algebra and to conjugacy by invertible G-fixed elements of that algebra (we will be talking abusively of "*the* Auslander-Reiten system ending with i"); our construction of that system proceeds mainly like the construction of the almost split sequence terminating in a given kG-module M (that is we take the pullback of a projective cover of ΩM over a generator of the $\mathrm{End}_{kG}(M)$-socle of $\mathrm{Ext}^1(M, \Omega^2 M)$).

Let k be the trivial kG-module and denote by $\mathcal{R}_k$ the almost split sequence terminating in k, and by $\mathcal{L}_k$ the corresponding Auslander-Reiten system of G. In [1], Auslander and Carlson show that the tensor product of any indecomposable kG-module M with the sequence $\mathcal{R}_k$ is either split or almost split up to an injective factor, and they give various criterions describing the second case (3.6., 4.7.). Using a different approach, we investigate a similar question for Auslander-Reiten systems : we give here sufficient conditions for certain pullbacks from the tensor product $i \otimes \mathcal{L}_k$ to be either split, or else equal up to a trivial system to the Auslander-Reiten system ending with i. In other words, denoting by u_G a representative of a generator of the socle of $\mathrm{Ext}^1(k, \Omega^2 k)$ (see §1), we look for cases when the equivalence class of some tensor product with u_G lies in the socle of the bimodule which corresponds here to $\mathrm{Ext}^1(M, \Omega^2 M)$. Since our sufficient conditions are not satisfied for all kG-modules, we obtain the related results of [1] (3.6, 4.7.) for very specific modules only. Yet our approach applies when i is a source idempotent of a block, and in that significant case (see [**2**], VII) we obtain an explicit generator of the socle. This last result, which was the starting point of this study, was suggested to me by Lluis Puig.

Section 1 presents our notations, the main definitions of **[2]** which are of use here, as well as two preliminary lemmas. We give our results in section 2.

§1 Notations and Preliminaries

We write $A^\times$ for the group of units of A and denote by a^x the element $\phi(x^{-1})a\phi(x)$ of A, where $a \in A$ and $x \in G$. The corresponding action of G makes A into a G-algebra. If H and K are two subgroups of G with $K \subset H$, we denote by A^H the algebra of H-fixed elements of A, and by $\mathrm{Tr}_K^H\colon A^K \to A^H$ the relative trace map, defined by $\mathrm{Tr}_K^H(a) = \sum a^x$, where x runs over a right transversal of H modulo K; the image of this map is the two-sided ideal A_K^H of A^H. If P is any p-subgroup of G, we denote by $Br_P : A^P \to A(P)$ the Brauer morphism, that is the homomorphism corresponding to the quotient by the ideal $\sum_{Q \subsetneq P} A_Q^P$ of A^P. Furthermore we set $\overline{A^H} = A^H / A_1^H$, and for any a in A^H we denote by $\overline{a}$ the image of a in $\overline{A^H}$. All modules and algebras are finite dimensional k-spaces, and the modules are left modules. We denote by A^{op} the opposite algebra of A, by $J(A)$ the Jacobson radical of A, and by ΠM (resp. ΩM) a projective cover (resp. a Heller translate) of the module M. Our tensor products are taken over k; note that the tensor product of two (symmetric) interior G-algebras is again a (symmetric) interior G-algebra.

Suppose we are given three mutually orthogonal idempotents i, i° and i' of A^G, together with two elements $d \in iA^G i^\circ$ and $d' \in i^\circ A^G i'$: we say that $\mathcal{S} = (i, i^\circ, i', d, d')$ is a *system* of G over A if we have $dd' = 0$ and if there exists (s, s') in $i^\circ A i \times i' A i^\circ$ satisfying the conditions :

$$i = ds\,, \quad i^\circ = sd + d's' \quad \text{and} \quad i' = s'd'\,.$$

In case $i^\circ \in A_1^G$, we call $\mathcal{S}$ a *Heller system*; we say $\mathcal{S}$ is *trivial* if $i = 0$, and *split* if $i \in dA^G$. Set $i^+ = i + i^\circ + i'$. The *commuting algebra* of the system $\mathcal{S}$ is by definition the interior G-subalgebra $A_\mathcal{S}$ of $i^+ A i^+$ whose elements commute with i, i°, i', d and d' simultaneously. We call $\mathcal{S}$ an *Auslander-Reiten system* if it is a non trivial system, if the algebra $A_\mathcal{S}^G$ is local, and if for every symmetric interior G-algebra B and every embedding of interior G-algebras $f\colon A \to B$, we have $f(d)B^G = f(i)J(B^G)$ (by definition an embedding $f\colon A \to B$ is a homomorphism of interior G-algebras that it is one-to-one and satisfies $\mathrm{Im}\, f = f(1)Bf(1)$). The idempotent i is then primitive in A^G (cf. **[2]**).

The following additional notations are fixed throughout this note. Considering projective covers of the kG-modules k and Ωk, we denote by E the interior G-algebra $\mathrm{End}_k(k \oplus \Pi k \oplus \Omega k \oplus \Pi(\Omega k) \oplus \Omega^2 k)$ and write e, e°, e', e'° and e'' for the orthogonal idempotents of E corresponding to the projections on k, Πk, Ωk, $\Pi(\Omega k)$ and $\Omega^2 k$ respectively. We consider Heller systems $\mathcal{H}_k = (e, e^\circ, e', h, h')$ and $\mathcal{H}_{\Omega k} = (e', e'^\circ, e'', m, m')$ of G over E (cf. **[2]**), and denote by u_G an element of $e'E^G e$ whose class $\overline{u_G}$ generates the 1-dimensional k-space $\overline{e'E^G e}$ (cf. **[2]**, III 3.).

On the other hand, we denote by P a Sylow p-subgroup of G, and we write $A(G)$ for the quotient of A^G by its two-sided ideal $\sum_{Q \subsetneq P} A_Q^G = \mathrm{Tr}_P^G(\ker Br_P)$ (if G is a p-group, then $P = G$ and we have $A(G) = Br_G(A^G)$, so the notation is consistent.) We begin with two lemmas which do not require A to be symmetric:

Lemma 1. *For any element a in $\sum_{Q \subsetneq P} A_Q^G$, we have $\overline{a \otimes u_G} = 0$.*

Proof: Let Q be a proper subgroup of P. We have $u_G \in E_1^Q \cap E^G$ (cf. [**2**], IV 2.1.), so every $a' \in A^Q$ satisfies $\mathrm{Tr}_Q^G(a') \otimes u_G \in (A \otimes E)_1^G$.

The converse of lemma 1 is true under certain conditions :

Lemma 2. *Suppose that $G = P$ and that the interior P-algebra A has a P-stable basis. Take u in E^P such that $\overline{u} \neq 0$ and let a be an element in A^P such that $Br_P(a) \neq 0$. Then $\overline{a \otimes u} \neq 0$.*

Proof: Set $v = a \otimes u$ and let $\mathcal{B}$ be a P-stable basis of A. The condition $Br_P(a) \neq 0$ ensures the existence of b_0 in $\mathcal{B} \cap A^P$ such that a has a non zero b_0-coordinate (cf. [**4**], 2.8.4.). We consider the projection of v onto $b_0 \otimes E$, in the decomposition $\bigoplus_{t \in \mathcal{B}} t \otimes E$ of $A \otimes E$; since u is not in E_1^P, we get $v \notin (A \otimes E)_1^P$.

§2

Fix an idempotent i in A^G, and set $B = iAi \otimes E$, $I = \sum_{Q \subsetneq P} iA_Q^G i$. The tensor product with e defines an embedding of interior G-algebras from iAi to B, and B is symmetric. Denote by $\mathcal{H} = (j, j^\circ, j', c, c')$ the Heller system $i \otimes \mathcal{H}_k$ of G over B. We recall from [**2**] that the $(\overline{j'B^Gj'}; \overline{jB^Gj})$-bimodule $\overline{j'B^Gj}$ has same socles as a left and as a right module, and that if i is primitive the socles have dimension 1 (cf. [**2**], III); furthermore in this case, if u is any element in $j'B^Gj$ whose class $\overline{u}$ generates that socle, the Auslander-Reiten system ending with i is equal, up to a trivial system and to embedding, to the pullback of the Heller system $i \otimes \mathcal{H}_{\Omega k}$ over (j, u) (cf. [**2**], VI).

Lemma 1 shows in particular that the condition $i \in I$ (which in case i is primitive means that the Sylow subgroup P is not a defect group of i), implies that the tensor product with i, or with any idempotent of iA^Gi, of the Auslander-Reiten system ending with e, is a split system. From now on we assume that the idempotent i does not belong to the ideal I.

Proposition 1. *For all a in iA^Gi whose class modulo I is in the $(iAi)(G)^{op}$-socle of $(iAi)(G)$, the element $\overline{a \otimes u_G}$ is in the socle of $\overline{j'B^Gj}$.*

Proof: Let a be such an element and set $u = a \otimes u_G$. Let $\hat{B}_{\mathcal{H}}$ denote the Heller algebra $B_{\mathcal{H}} \oplus j'Bj$ of $\mathcal{H}$ (cf. [**2**]). In [**2**], III 3. we show that every "symmetrising" form τ on B determines a central form $\tau_{\mathcal{H},G}$ on $\hat{B}_{\mathcal{H}}^G$, which annihilates $(\hat{B}_{\mathcal{H}})_1^G$ and induces a symmetrising form on $\overline{\hat{B}_{\mathcal{H}}^G}$; moreover the socle of $\overline{j'B^Gj}$ coincides with the orthogonal of the radical of $\overline{\hat{B}_{\mathcal{H}}^G}$. Thus it is sufficient to prove that $\tau_{\mathcal{H},G}(u \cdot J(\hat{B}_{\mathcal{H}}^G)) = 0$. Since we have $u \cdot J(\hat{B}_{\mathcal{H}}^G) = uJ((iAi \otimes eEe)^G)$ and $eEe = (eEe)^G \simeq k$, all we need to show is that the restriction of the form $\tau_{\mathcal{H},G}$ to the space $(aJ(iA^Gi)) \otimes u_G$ is zero. But the hypothesis yields $aJ(iA^Gi) \in I$, so we conclude by lemma 1.

Corollary. *Suppose we have $J(iA^Gi) = I$. Then for all a in iA^Gi the element $\overline{a \otimes u_G}$ lies in the socle of $\overline{j'B^Gj}$.*

Proof: In this case the radical of $(iAi)(G)$ is $\{0\}$.

Let us assume next that $G = P$ and that i is primitive. Thus the algebra $(iAi)(P)$ is local, and it has a right socle of dimension 1. In certain cases we obtain a generator of the socle of $\overline{j'B^Pj}$:

Proposition 2. *Suppose that the P-algebra iAi has a P-stable basis. Let a be an element of iA^Pi whose image under Br_P generates the $(iAi)(P)^{op}$-socle of $(iAi)(P)$. The socle of $\overline{j'B^Pj}$ is the k-space generated by $\overline{a \otimes u_P}$.*

Proof: Lemma 2 shows that $\overline{a \otimes u_P} \neq 0$. The result then follows from proposition 1.

Remark: (Application to kG-modules) Let us consider the special case where A is the algebra of k-endomorphisms of an indecomposable kG-module M with vertex P, and where $i = \mathrm{id}_M$. If M is simple, or if $G = P$ and M is an endo-permutation kP-module, then $(iAi)(G) \simeq k$ (cf. [**5**], 5.8.), so the corollary applies: for $a = i$, our statement says that if M is simple, the tensor product of the almost split sequence $\mathcal{R}_k$ with M is either split, or almost split up to a projective direct summand. On the other hand if M is an endo-permutation kP-module, proposition 2 shows that this same tensor product is, up to a projective direct summand, the almost split sequence terminating in M. These are two special cases of the results of Auslander and Carlson on the tensor product of the sequence $\mathcal{R}_k$ with a kG-module, cf. [**1**], 3.6., 4.7. (indeed if M is an endo-permutation module, we have $p \nmid \dim M$, because every P-stable basis of $iAi = \mathrm{End}_k(M)$ contains a unique fixed point, cf. [**4**], 2.8.4.).

Application to the source algebra of a block. Let b be a primitive idempotent of the center ZkG of kG, and set $A = kGb$. Assume b has a non trivial defect group, say D, and let i be a D-source of b, that is a primitive idempotent of A^D such that $b \in \mathrm{Tr}_D^G(A^D iA^D)$. The hypotheses of proposition 2 are satisfied for the source algebra iAi and $P = D$. Let $\mathcal{S}Z(D)$ denote the element $\sum_{x \in Z(D)} x$ of kD. We obtain an explicit generator of the socle of $\overline{j'B^Dj}$:

Theorem. *Set $a = \mathcal{S}Z(D) \cdot i$. The socle of $\overline{j'B^Dj}$ is the k-vector space generated by $\overline{a \otimes u_D}$.*

Proof: Viewing the isomorphism of interior $Z(D)$-algebras $(iAi)(D) \simeq kZ(D)$ (cf. [**4**], 14.5.) as an identification, the Brauer morphism $Br_D : iA^Di \to (iAi)(D)$ maps a to the element $\mathcal{S}Z(D)$ of $kZ(D)$. Thus the $kZ(D)^{op}$-module generated by $Br_D(a)$ is trivial, that is isomorphic to k and equal to the $kZ(D)^{op}$-socle of $kZ(D)$. The conclusion now follows from proposition 2.

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Odile GAROTTA
Institut Fourier
BP 74
F–38402 SAINT-MARTIN-D'HÈRES Cedex