G. R. Robinson
R. Staszewski

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Astérisque, tome 181-182 (1990), p. 237-255
<http://www.numdam.org/item?id=AST_1990__181-182__237_0>

MORE ON ALPERIN'S CONJECTURE

by G. R. ROBINSON AND R. STASZEWSKI

INTRODUCTION: We assume that the reader is familiar with the results, notation, and methods of [5]. Results from that paper (and minor variants thereof) will sometimes be quoted without explicit reference.

Our main aim in this paper is to try to understand the relevance of Clifford - theoretic techniques to Alperin's conjecture. Thus we are concerned with the effect of the presence of normal subgroups when trying to prove Alperin's conjecture. Once more, we try to maintain the dual viewpoint of applying our results to groups for which Alperin's conjecture is known to be valid, and trying to prove the conjecture in general (or to at least obtain some control of the structure of a minimal counterexample). Thus, for example, all the results of the first section are valid for p-solvable groups, since Alperin's conjecture is known to be valid for p-solvable groups (for the prime p), (see Okuyama [7]).

The main result of the first section is that Alperin's conjecture is equivalent to an apparently stronger conjecture which seems more compatible with the presence of normal subgroups.

In the second section, we prove that a minimal counterexample to Alperin's conjecture (for the prime p) has no normal subgroup of index p.

In the third section, we propose an "equivariant" form of Alperin's conjecture, that is a form of Alperin's conjecture which predicts compatibility with the action of a group of automorphisms. As far as we can tell at present, this conjecture is genuinely stronger than Alperin's conjecture.

In his Arcata article [1], Alperin suggests that in trying to prove his conjecture, other, more general, conjectures might naturally arise and need to be proved along the way. We believe that the results and methods of this article are the beginnings of a fulfilment of that prediction.

NOTATION: Throughout, p denotes a fixed prime, and k is the algebraic closure of GF(p). When dealing with the complexes $\mathcal{P}, \mathcal{N}, \mathcal{E}, \mathcal{U}$ of [5], it will sometimes be necessary to indicate the group from which the subgroups involved are taken, so we may speak of $\mathcal{P}(G)$, etc.

When Q is a p-subgroup of the finite group G, and B is a sum of blocks of kG, we let $f_0^{(B)}(N_G(Q)/_Q)$ denote the number of (isomorphism types of) projective simple $kN_G(Q)/_Q$-modules in Brauer correspondents of B.

SECTION ONE: THE CONJECTURE & NORMAL SUBGROUPS.

Let G be a finite group for which every proper section of G satisfies Alperin's conjecture for the prime p (for every p-block). Let $H(\neq G)$ be a normal subgroup of G, and let B be a block of kG which does not lie over blocks of defect 0 of kH, say B lies over the block b of kH.

The following lemma is well-known:

LEMMA 1.1: Let X be a finite group, B be a block of kX with defect group D. Let Q be a subgroup of D such that $kN_X(Q)/_Q$ has a projective simple module, S say, which lies in a Brauer correspondent of B. Then there is a conjugate, Q_1, of Q such that $Q_1 \subset D$ and $C_D(Q_1) \subset Q_1$.

<u>PROOF</u>: Let b be the block of $kN_X(Q)$ containing S. Then S lies over a projective simple $kQC_X(Q)/_Q$-module, so b lies over a block of defect Z(Q) of $k\,C_X(Q)$, say b*. Then $(Q,b^*) \supset (1,B)$. Let (D_1,b') be a maximal B-subpair with $(Q,b^*) \subset (D_1,b')$. We claim that $C_{D_1}(Q) \subset Q$. Otherwise, there is a B-subpair $(QC_{D_1}(Q),b'')$ with $(Q,b^*) \subsetneq (QC_{D_1}(Q),b'') \subset (D_1,b')$, contrary to the fact that b* is a block of defect Z(Q) of $C_X(Q)$. Thus $C_{D_1}(Q) \subset Q$, and the result follows as $D_1 = D^x$ for some $x \in X$.

COROLLARY 1.2: $\sum_{C\in\mathcal{P}(G)/_G} (-1)^{|C|}\, \ell(B_C) = \sum_{C\in\mathcal{P}(H)/_G} (-1)^{|C|}\, \ell(B_C)$.

<u>PROOF</u>: Let D be a defect group for B. Then $D\cap H \neq 1_G$, as B does not lie over blocks of defect 0 of kH. Thus $Z(D) \cap H \neq 1_G$, as $D\cap H \lhd D$.

Hence whenever Q is a p-subgroup of G with $Q\cap H = 1_G$, we have $f_0^{(B)}(N_G(Q)/_Q) = 0$ (for when Q_1 is a conjugate of Q with $Q_1 \subseteq D$, we have $Z(D) \cap H \subset C_D(Q_1)$, but $Z(D) \cap H \not\subseteq Q_1$).

Now $\sum_{C\in\mathcal{P}(G)/_G} (-1)^{|C|}\, \ell(B_C) = \sum_{C\in\mathcal{N}(G)/_G} (-1)^{|C|}\, \ell(B_C)$, and

similarly $\sum_{C\in\mathcal{P}(H)/_G} (-1)^{|C|}\, \ell(B_C) = \sum_{C\in\mathcal{N}(H)/_G} (-1)^{|C|}\, \ell(B_C)$. By an argument similar to that of Proposition 3.3 of [5] we may pair off orbits of chains of $\mathcal{N}(G)\backslash\mathcal{N}(H)$ whose first non-trivial term meets H non-trivially (that is to say given such a chain $C = Q_0 < Q_1 < \ldots < Q_n$ with

$Q_1 \cap H \neq 1_G$ and $Q_n \not\leq H$, form the chain C' as follows: let Q_i be the first term of C for which $Q_i \not\leq H$. If $Q_{i-1} = Q_i \cap H$, delete Q_{i-1} from C, whilst if $Q_{i-1} \neq Q_i \cap H$, insert $Q_i \cap H$ into C (between Q_{i-1} and Q_i). Then we pair the orbit of C with that of C').

We thus obtain:

$$\sum_{C \in \mathcal{N}(G)/G} (-1)^{|C|} \ell(B_C) = \sum_{C \in \mathcal{N}(H)/G} (-1)^{|C|} \ell(B_C) + \sum_{(Q)} \sum_{\{\text{chains } C \text{ whose first non-trivial term is } Q\}/N_G(Q)} (-1)^{|C|} \ell(B_C)$$

where (Q) runs over a set of representatives for the conjugacy classes of p-subgroups Q, of G with $Q \cap H = 1_G \neq Q$.

Since Alperin's conjecture holds within proper sections of G, we obtain

$$\sum_{C \in \mathcal{P}(G)/G} (-1)^{|C|} \ell(B_C) = \sum_{C \in \mathcal{P}(H)/G} (-1)^{|C|} \ell(B_C) - \sum_{(Q) \text{ as above}} f_o^{(B)}(N_G(Q)/Q).$$

Since $f_o^{(B)}(N_G(Q)/Q) = 0$ whenever $Q \cap H = 1_G$, the result follows.

COROLLARY 1.3: We have

$$\sum_{C \in \mathcal{P}(H)/G} (-1)^{|C|} \ell(B_C) = \ell(B) - \sum_{(Q)} \sum_{(b')} \ell(b')$$

where Q runs over p-subgroups of H (up to G-conjugacy) and b' runs over blocks of $N_G(Q)/Q$, with $b'^G = B$, lying over blocks of defect 0 of $kN_H(Q)/Q$.

<u>PROOF</u>: Let Q be a non-trivial p-subgroup of H. Then Alperin's conjecture holds for blocks of $kN_G(Q)/Q$. Let $X = N_G(Q)/Q$, $Y = N_H(Q)/Q$.

Let b' be a block of kX with $b'^G = B$. Suppose that b' does <u>not</u> lie over blocks of defect 0 of kY. Then, as in Corollary 1.2, we obtain

$$\sum_{C \in \mathcal{N}(X)/X} (-1)^{|C|} \ell(b'_C) = \sum_{C \in \mathcal{N}(Y)/X} (-1)^{|C|} \ell(b'_C)$$

On the other hand, b' is of Lefschetz type, since b' has positive defect and Alperin's conjecture holds for all blocks of all proper sections of G. Thus $\sum_{C \in \mathcal{N}(Y)/X} (-1)^{|C|} \ell(b'_C) = 0$.

If b' <u>does</u> lie over blocks of defect 0 of kY, then it is clear that $\sum_{C \in \mathcal{N}(Y)/X} (-1)^{|C|} \ell(b'_C) = \ell(b')$, since $b'_C = 0$ unless C is the chain $\{1_Y\}$.

It follows, then, that the contribution to $\sum_{C \in \mathcal{N}(H)/_G} (-1)^{|C|} \ell(B_C)$ from chains whose first non-trivial term is (G-conjugate to) Q is $-\sum_{b'} \ell(b')$, where b' runs over blocks of $kN_G(Q)/_Q$ with $b'^G = B$, lying over blocks of defect 0 of $kN_H(Q)/_Q$. The result now follows.

COROLLARY 1.4 : B is of Lefschetz type if and only if $\ell(B) = \sum_{(Q)} \sum_{(b')} \ell(b')$, where Q runs over p-subgroups of H (up to G-conjugacy), and b' runs over blocks of $kN_G(Q)/_Q$ with $b'^G = B$ lying over blocks of defect 0 of $kN_H(Q)/_Q$.

We can now state:

PROPOSITION 1.5 (ANOTHER FORMULATION OF ALPERIN'S CONJECTURE): The following are equivalent:

i) Whenever X is a finite group, and B is a block of kX, B is of Alperin type.

ii) Whenever X is a finite group, $Y \lhd X$ and B is a block of kX we have $\ell(B) = \sum_{(Q)} \sum_{(b')} \ell(b')$, where Q runs over p-subgroups of Y up to X-conjugacy, and b' runs over blocks of $kN_X(Q)/_Q$, with $b'^X = B$, lying over blocks of defect 0 of $kN_Y(Q)/_Q$.

PROOF : It is clear that ii) implies i) (taking Y = X). The results of this section show that i) implies ii) (upon noting that ii) holds vacuously if B lies over blocks of defect 0 of kY).

Now suppose that b has defect group P. Then there is a unique block, B*, of $kN_G(P)$ with $B^{*G} = B$ and B* lies over b* (the Brauer correspondent of b in $kN_H(P)$) (Harris-Knörr [4]).

COROLLARY 1.6 : Suppose that one of the following occurs:

i) b is nilpotent

ii) P is Abelian.

iii) $P \cap P^h = 1_H$ for all $h \in H \backslash N_H(P)$.

Then B is of Lefschetz type if and only if $\ell(B) = \ell(B^*)$.

PROOF : By assumption, Alperin's conjecture holds within H. Thus $f_o^{(b)}(N_H(P)/_P) = \ell(b)$ and $f_o^{(b)}(N_H(Q)/_Q) = 0$ whenever Q is a p-subgroup of H not H-conjugate to P in any of the cases listed (see [1]). The same applies to any G-conjugate of b (using the appropriate G-conjugate of P). It readily follows that the formula appearing in the statement of Corollary 1.4 reduces to : $\ell(B) = \ell(B^*)$ in the cases listed.