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A Mackey Functor Version of a Conjecture of Alperin

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The conjecture of Alperin we consider has to do with modular representations of a finite group G over an algebraically closed field k of characteristic p.

CONJECTURE 1 (Alperin [2]). *The number of weights for G equals the number of simple kG-modules.*

Alperin makes his definition of a *weight* for G in [2]. There now exist various equivalent forms of this conjecture, and we will work with the one which appeared first after Alperin's original version. We let $\mathrm{np}(G)$ denote the number of non-projective simple kG-modules, and Δ the simplicial complex of chains of non-identity p-subgroups of G (see [10]).

CONJECTURE 2 (Knörr–Robinson [5]). *For all finite groups G,*

$$\mathrm{np}(G) = \sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma}\, \mathrm{np}(G_\sigma).$$

The notation $\mathrm{np}(G)$ differs from what appears in print elsewhere. In the notation of [5] and [10] one has $\mathrm{np}(G) = \ell(G) - f_0(G)$ where $\ell(G)$ is the number of simple kG-modules and $f_0(G)$ is the number of blocks of defect zero. In comparing different printed versions it may help to notice that for the stabilizer groups G_σ we have $\mathrm{np}(G_\sigma) = \ell(G_\sigma)$ because the G_σ always have a non-trivial normal p-subgroup, and so $f_0(G_\sigma) = 0$.

We will show that Conjectures 1 and 2 are equivalent to the next conjecture.

CONJECTURE 3. *For every finite group G, for every prime p, there exist Mackey functors M_1, M_2 so that $M_1(H)$ and $M_2(H)$ are vector spaces over a field R whose characteristic is 0 or prime to $|G|$, satisfying*

(i) For every subgroup H, the restrictions $M_1 \downarrow_H^G$ and $M_2 \downarrow_H^G$ are projective relative to p-local subgroups of H.

(ii) For every subgroup H, $\dim M_1(H) - \dim M_2(H) = \mathrm{np}(H)$.

THEOREM 4. *Conjectures 2 and 3 are equivalent.*

The interesting thing about this conjecture is that somehow it conveys the information of Alperin's conjecture in the structure of the Mackey functors. In other forms of the conjecture there is a sum over some reasonably complicated indexing set, be it conjugacy classes of p-subgroups as in Alperin's original version, or conjugacy classes of chains of p-subgroups as in Robinson's version. In Conjecture 3 there is no such sum. The hope is that because arithmetic is somehow replaced by structure there may come about an understanding of what is going on. With the exception of the groups of Lie type in defining characteristic p, it seems that every verification of Alperin's conjecture for a particular class of groups has been an observation of a numerical equality. The conjecture may eventually be completely proved in such a fashion, but there will still be the need for an explanation, and here the Mackey functors may come in.

In this note we first prove Theorem 4, which relies on the theorem in [11] combined with a description of the irreducible Mackey functors [9]. Someone familiar with [11] will quickly see the connection between Conjecture 3 and Conjecture 2, and in fact much of the motivation behind [11] came from trying to prove Alperin's conjecture in this way. From this point of view we are really interested in the implication that Conjecture 3 implies Conjecture 2. For the proof of the converse statement we have to construct Mackey functors M_1 and M_2 satisfying the conditions of Conjecture 3, but the construction we have is rather artificial. Our interest in this implication is the consequence that so long as one believes that Alperin's conjecture is true then it is not a waste of time to study Conjecture 3. It seems probable that if one can prove Alperin's conjecture using Mackey functors then it will be done by some natural construction and artificial constructions will not do.

Conjecture 3 arose out of a less complicated set of conditions, which unfortunately turned out to have a counterexample. We present this set of conditions as Question 13, and give the counterexample after that.

We conclude in the last section by presenting the most general situation in which we have constructed Mackey functors satisfying Conjecture 3. This is the case of groups which have a cyclic Sylow p-subgroup, and we use the existing theory of modular representations of such groups to construct the Mackey functors. It is exactly this theory which can be used directly to establish Alperin's Conjecture and so we should make it clear that we have no new approach here. The interest in our construction of these Mackey functors is that it may give some indication of what one should expect in general.

1. Preliminaries on Mackey functors

The reader should refer to [3], [4] and [11] for the definition of a Mackey functor and the notion of relative projectivity. We will regard a Mackey functor as being defined on the set of all subgroups of G (rather than on G-sets, which is the approach taken in [3] and [4]). We write induction, restriction and conjugation mappings as I_H^G, R_H^G and c_g.

We will use the operations on Mackey functors of *restriction, induction* and *inflation.* When N is a Mackey functor for G and $H \leq G$ we obtain a Mackey functor $N \downarrow_H^G$ on H by restricting attention to subgroups of H. Thus $N \downarrow_H^G (K) = N(K)$. Induction first appears in [6], where it is attributed to Yoshida. Suppose M is a Mackey functor defined for a subgroup H of G. Using temporarily the G-set notation, we define $M \uparrow_H^G$ to be the Mackey functor for G which is given by $M \uparrow_H^G (\Omega) = M(\Omega \downarrow_H^G)$, where Ω is a G-set and $\Omega \downarrow_H^G$ is its restriction as an H-set. In subgroup notation this becomes

$$M \uparrow_H^G (K) = \bigoplus_{x \in H\backslash G/K} M(H \cap {}^xK), \qquad K \leq G.$$

We now define the notion of inflation. Suppose that G has a normal subgroup N with $G/N = Q$ and M is a Mackey functor defined on the subgroups of Q. We construct a Mackey functor $\operatorname{Inf}_Q^G M$ defined on subgroups K of G by

$$\operatorname{Inf}_Q^G M(K) = \begin{cases} 0 \text{ if } K \not\geq N \\ M(K/N) \text{ if } K \geq N. \end{cases}$$

Here K/N is a subgroup of Q. Restriction, induction and conjugation mappings are necessarily zero except between subgroups containing N, when they are defined to be the mappings $I_{H/N}^{K/N}, R_{H/N}^{K/N}, c_{gN}$ with $H, K \geq N, g \in G$. The fact that one extends M by zero on subgroups not containing N is at first a surprise, but it is the canonical way to make such an extension.

PROPOSITION 5. *Let H be a subgroup of G, M a Mackey functor for $N_G(H)/H$. Put $L = (\operatorname{Inf}_{N_G(H)/H}^{N_G(H)} M) \uparrow_{N_G(H)}^G$. Then*
(i) $L(K) = 0$ unless K contains a conjugate of H,
(ii) $L(H) = M(1)$.

Proof. From the definition,

$$L(K) = \bigoplus_{g \in N_G(H)\backslash G/K} \operatorname{Inf}_{N_G(H)/H}^{N_G(H)} M(N_G(H) \cap {}^gK).$$

The terms are zero unless ${}^gK \geq H$, whence condition (i). Furthermore, when $K = H$ we only get a non-zero contribution when ${}^gH \geq H$, but this means $g \in N_G(H)$ so there is only one such term and it occurs with $g = 1$.

COROLLARY 6. *Given any subgroup $H \leq G$ there exists a Mackey functor L on G such that*
(i) $L(H)$ is a one-dimensional vector space over a field R,
(ii) $L(K) = 0$ unless $H \leq {}^gK$ for some $g \in G$,
(iii) If $O_p(H) \neq 1$ then for all subgroups K of G, $L \downarrow_K^G$ is projective relative to p-local subgroups of K.

Proof. To prove (i) and (ii), apply Proposition 5 with M the Mackey functor on $N_G(H)/H$ which has $M(K/H) = R$ for all subgroups $K \geq H$. Restriction and conjugation mappings are the identity and induction mappings are the multiplication by the index.

For (iii) we use the formula

$$L \downarrow_K^G = (\operatorname{Inf} M) \uparrow_{N_G(H)}^G \downarrow_K^G = \bigoplus_{g \in K\backslash G/N_G(H)} c_g((\operatorname{Inf} M) \downarrow_{N_G(H)\cap {}^{g^{-1}}K}^{N_G(H)}) \uparrow_{{}^gN_G(H)\cap K}^K .$$

Each summand is zero unless ${}^{g^{-1}}K \supseteq H$, i.e. $K \supseteq {}^gH$, in which case it is induced from ${}^gN_G(H) \cap K$, which has a normal p-subgroup $O_p({}^gH)$. Thus each non-zero term is also induced from the larger subgroup $N_K(O_p({}^gH))$, which is a p-local subgroup of K. Hence the result.

We need to quote a description of the irreducible Mackey functors which we do not prove here. We state the result in the case of Mackey functors over R where R is a field of characteristic 0 or prime to $|G|$. This means Mackey functors M for which $M(H)$ is always a vector space over R. This is a special case of a more general result without this restriction on R which is proved in [9]. As is usual in an abelian category, a Mackey functor is said to be *irreducible* if it has no non-trivial proper subfunctors.

THEOREM 7. *Let R be a field whose characteristic is either 0 or prime to $|G|$. Up to isomorphism, the irreducible Mackey functors over R biject with pairs (H, V) where H is a subgroup of G determined up to conjugacy and V is an irreducible $RN_G(H)/H$-module. The irreducible Mackey functor $S_{H,V}$ corresponding to such a pair is constructed as follows. Let M be the Mackey functor on $N_G(H)/H$ given by $M(K) = V^K$ for $K \leq N_G(H)/H$. Then $S_{H,V} = (\operatorname{Inf}_{N_G(H)/H}^{N_G(H)} M) \uparrow_{N_G(H)}^G$.*

We will also need to quote the following result, which is obtained by combining Theorem 7 with 12.2 and 12.3 of [7] (see also [9]).

THEOREM 8. *Let M be a Mackey functor over R, where R is a field whose characteristic is 0 or prime to $|G|$. Then M is a direct sum of irreducible Mackey functors.*