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A Canonical Brauer Induction Formula

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Introduction

Throughout this paper G denotes a finite group, $R(G)$ the character ring of G and $(-,-)$ the usual inner product of $R(G)$.
In 1946 Richard Brauer proved (cf. [Brl]) that each virtual character χ of G can be expressed as a linear combination

$$\chi = \sum_i z_i \mathrm{ind}_{H_i}^G \varphi_i$$

where $z_i \in \mathbf{Z}$, $H_i \leq G$ and $\varphi_i \in \hat{H}_i = \mathrm{Hom}(H_i, \mathbf{C}^*)$. Brauer was motivated by the question whether Artin L-functions of any virtual character have a meromorphic extension to the entire complex plane. This was known for one-dimensional characters, and it was also known that the Artin L-functions are invariant under induction. So Brauer's induction theorem gave a positive answer to the above question, and this is a very typical example for the applications of the theorem in number theory. However, Brauer's theorem is a mere existence theorem, and it remained the question for an explicit formula, associating to each virtual character χ an integral linear combination as above. A first result in this direction is again due to Brauer, who gave in 1951, cf. [Br2], an explicit formula to Artin's induction theorem, i.e. a formula which induces from cyclic subgroups and has rational coefficients. It was not before 1986, that there appeared V. Snaith's explicit version of Brauer's induction theorem, cf. [Sn]. His formula is based on topological invariants, in particular on Euler characteristics of quotient spaces of the unitary group $U(n)$.
Here we give an explicit and canonical formula for Brauer's induction theorem by algebraic and combinatorial methods. 'Canonical' means that this formula is unique among all the expressions for χ as above, if a certain functorial behaviour with respect to G is required. To state this functorial property it is convenient to introduce the free abelian group $R_+(G)$ whose basis is

given by the G-conjugacy classes of pairs (H,φ), where $H \leq G$ and $\varphi \in \hat{H}$, cf. [De], p. 11. We consider a formula as a map from $R(G)$ to $R_+(G)$, such that it becomes the identity, if the symbols (H,φ) are replaced by $\mathrm{ind}_H^G\varphi \in R(G)$. It turns out that $R_+(G)$ carries a lot of structures, which we investigate in section 1. Using the results about $R_+(G)$ we define the formula $a_G : R(G) \longrightarrow R_+(G)$ and prove its natural properties, cf. theorem (2.1) and cor. (2.12). In section 3 we apply the methods developed in the previous sections to obtain an induction formula which induces only from subgroups of a fixed type $\mathcal{T}$, cf. theorem (3.2). In this case however, we don't have integral coefficients any longer. For the type of cyclic groups we obtain again Brauer's explicit version [Br2] of Artin's induction theorem. The cases in which the formula is integral are determined in (3.12) and (3.13). Unfortunately the formula is not integral for the type of elementary groups. For the type of cyclic groups we obtain that the "worst" denominator in the formula for the characters of G coincides with the Artin exponent of G.

The formula a_G we introduce in section 2 is different from Snaith's formula in [Sn], but there is a relation between them which can be found in [Bo], chap. IV.

I am grateful to G.-M. Cram for his proof of proposition (2.24).

1. The ring $R_+(G)$

For a finite group G we consider the set $\mathcal{M}_G$ of all pairs (H,φ) where $H \leq G$ and $\varphi \in \hat{H} = \mathrm{Hom}(H, \mathbf{C}^*)$. G acts from the left on $\mathcal{M}_G$ by componentwise conjugation: ${}^g(H,\varphi) := ({}^gH, {}^g\varphi)$ where ${}^gH = gHg^{-1}$, ${}^g\varphi := \varphi(g^{-1}.g)$, for $g \in G$. We denote the G-orbit of (H,φ) by $\overline{(H,\varphi)}^G$ and the set of G-orbits by $\mathcal{M}_G/G$. Let $R_+(G)$ be the free abelian group with the basis $\mathcal{M}_G/G$, then we have the well-defined map into the character ring $R(G)$

$$(1.1) \qquad b_G : R_+(G) \longrightarrow R(G), \quad \overline{(H,\varphi)}^G \mapsto \mathrm{ind}_H^G\varphi.$$

b_G is surjective by Brauer's induction theorem [Br1]. We want to construct a map

$$(1.2) \qquad a_G : R(G) \longrightarrow R_+(G), \quad \chi \mapsto \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi)\overline{(H,\varphi)}^G$$

with $b_G a_G = \mathrm{id}_{R(G)}$, i.e.

$$(1.3) \qquad \chi = \sum_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G} \alpha_{\overline{(H,\varphi)}^G}(\chi)\mathrm{ind}_G^H\varphi$$

for all $\chi \in R(G)$. Moreover we want a_G to have a good functorial behaviour with respect to the structures carried by $R(G)$ and $R_+(G)$.

(1.4) Remark. We may consider $R_+(G)$ as the Grothendieck group of the category of *monomial representations of G*. Its objects are finite dimensional $\mathbf{C}G$-modules V ($\mathbf{C}G$ denotes the group ring) with a fixed decomposition $V = V_1 \oplus \ldots \oplus V_n$ into one-dimensional subspaces, called the *lines* of V, such that G permutes the lines. V is called *simple*, if its lines are permuted transitively by G. A *morphism* $F : V = V_1 \oplus \ldots \oplus V_n \longrightarrow W = W_1 \oplus \ldots \oplus W_m$ of two monomial representations of G is a $\mathbf{C}G$-linear map such that for each $i \in \{1, \ldots, n\}$ there is some $j \in \{1, \ldots, m\}$ with $F(V_i) \subseteq W_j$. For monomial representations we may define in an obvious way direct sums, tensor products, duals, restriction maps along group homomorphisms and induction maps along subgroup relations. Every monomial representation of G is a unique direct sum of simple ones and the isomorphism classes of simple monomial representations are in a bijective correspondence to $\mathcal{M}_G/G$ by the following construction: For simple $V = V_1 \oplus \ldots \oplus V_n$ define H to be the stabilizer of V_1 and $\varphi \in \hat{H}$ to be the action of H on V_1. The choice of another line V_i gives a conjugated pair ${}^g(H, \varphi)$. b_G is induced from the forgetful functor which associates to every monomial representation of G the underlying $\mathbf{C}G$-module. For more details of the above statements see [Bo] chap.I §1.

The constructions described above provide $R_+(G)$ with the following structures:

Multiplication. The tensor product on monomial representations is translated into a commutative ring structure on $R_+(G)$ given by

$$\overline{(H,\varphi)}^G \cdot \overline{(K,\psi)}^G = \sum_{s \in H\backslash G/K} \overline{(H \cap {}^sK, \varphi \cdot {}^s\psi)}^G. \tag{1.5}$$

The unity is $\overline{(G,1)}^G$. $R_+(G)$ contains the group ring $\mathbf{Z}\hat{G} \cong \oplus_{\varphi \in \hat{G}} \mathbf{Z}\overline{(G,\varphi)}^G$ as a subring. Note that the G-orbit of (G, φ) consist only of this single pair. So $R_+(G)$ is a $\mathbf{Z}\hat{G}$-algebra and b_G is a $\mathbf{Z}\hat{G}$-algebra map. We have the $\mathbf{Z}\hat{G}$-module decomposition

$$R_+(G) = \mathbf{Z}\hat{G} \oplus \bigoplus_{\overline{(H,\varphi)}^G \in \mathcal{M}_G/G, H<G} \mathbf{Z}\overline{(H,\varphi)}^G,$$

with the corresponding projection map

$$\pi_G : R_+(G) \longrightarrow \mathbf{Z}\hat{G}, \quad \overline{(H,\varphi)}^G \mapsto \begin{cases} \varphi, & \text{if } H = G; \\ 0, & \text{if } H < G. \end{cases} \tag{1.6}$$

Also $R(G)$ is a $\mathbf{Z}\hat{G}$-algebra, since it contains $\mathbf{Z}\hat{G}$ as a subring. This gives rise to the $\mathbf{Z}\hat{G}$-module decomposition

$$R(G) = \mathbf{Z}\hat{G} \oplus \bigoplus_{\chi \in \mathrm{Irr}G \setminus \hat{G}} \mathbf{Z}\chi$$

where $\mathrm{Irr}G$ is the set of irreducible characters of G. We obtain the corresponding projection

$$(1.7) \qquad p_G : R(G) \longrightarrow \mathbf{Z}\hat{G}, \quad \mathrm{Irr}G \ni \chi \mapsto \begin{cases} \chi, & \text{if } \chi \in \hat{G}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that π_G is multiplicative, which is in general not true for p_G.

Restriction. The restriction of monomial representations of G along a group homomorphism $f : G' \longrightarrow G$ gives rise to the ring homomorphism

$$(1.8) \qquad \mathrm{res}_{+f} : R_+(G) \longrightarrow R_+(G') : \quad \overline{(H,\varphi)}^G \mapsto \sum_{s \in f(G')\backslash G/H} \overline{(f^{-1}({}^sH), {}^s\varphi \circ f)}^{G'}.$$

The diagram

$$(1.9) \qquad \begin{array}{ccc} R_+(G) & \xrightarrow{b_G} & R(G) \\ {\scriptstyle \mathrm{res}_{+f}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_f} \\ R_+(G') & \xrightarrow{b_{G'}} & R(G') \end{array}$$

commutes, since the corresponding diagram on the level of the categories of (monomial) representations commutes. For the same reason we have

$$(1.10) \qquad \mathrm{res}_{+ff'} = \mathrm{res}_{+f'} \mathrm{res}_{+f}$$

for another group homomorphism $f : G'' \longrightarrow G'$. If f is given as the inclusion of a subgroup $H \leq G$, we write res_{+H}^G instead of res_{+f} and obtain from (1.8)

$$(1.11) \qquad \mathrm{res}_{+H}^G : R_+(G) \longrightarrow R_+(H), \quad \overline{(K,\psi)}^G \mapsto \sum_{s \in H\backslash G/K} \overline{(H \cap {}^sK, {}^s\psi)}^H.$$

If $f : G \longrightarrow G/N =: \overline{G}$ is the canonical surjection for a normal subgroup N of G, we obtain

$$(1.12) \qquad \mathrm{res}_{+f} \overline{(H/N, \overline{\varphi})}^{\overline{G}} = \overline{(H,\varphi)}^G$$

where $N \leq H \leq G$ and $\varphi \in \hat{H}$ vanishes on N. Thus res_{+f} maps the basis $\mathcal{M}_{\overline{G}}/\overline{G}$ injectively into the basis $\mathcal{M}_G/G$. We use the restriction maps to define the ring homomorphism

$$(1.13) \qquad \rho_G : R_+(G) \longrightarrow \prod_{H \leq G} \mathbf{Z}\hat{H}, \quad x \mapsto \left(\pi_H \mathrm{res}_{+H}^G x\right)_{H \leq G}.$$