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ON THE SPACE OF MAPS BETWEEN R-LOCAL CW COMPLEXES

by

D.J. Anick[1] and E. Dror Farjoun

1. Summary of Results and Notations

The papers [A1,A2] introduced and studied a differential graded Lie algebra (dgL) associated as a model to certain spaces. Building on that work, we construct in this note a simplicial skeleton for the space of pointed maps between two R-local simply-connected CW complexes ($R \subset \mathbf{Q}$). The construction entails two steps. First is the construction, in the category of dgL's, of a cosimplicial resolution and an associated "function complex" valid in a range of dimensions; and second is the connection with the topological mapping space via the above-mentioned models.

1.1. A function complex for dgL's. Let $R = \mathbf{Z}[(p-1)!]^{-1} \subset \mathbf{Q}$ for a prime p, and let L, M be free r-reduced dgL's over R having all generators in dimensions below rp ($r \geq 1$). We will construct a simplicial set, to be denoted <u>hom</u>(L,M), which serves in a range of dimensions as a function complex in the sense of Dwyer and Kan [DK]. Our construction is explicit, in terms of generators and differentials; it is something which could be implemented on a computer. When L and M arise as models for finite spaces X and Y, this means that a simplicial model for the pointed mapping space Y^X is computable in a range of dimensions.

1.2. The range of dimensions. When X and Y are R-local r-connected CW complexes ($r \geq 1$), whose dimensions m_X and m_Y are bounded above by m and by rp respectively ($m < rp$), we may associate to them the dgL models L_X and L_Y. Then Y^X has the d-type of $\mathrm{hom}(L_X, L_Y)$, where

$$d = \min(rp - 1,\ r + 2p - 3) - m\ .$$

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Beyond dimension d, <u>hom</u>(L_X, L_Y) is still defined, but its connection with the geometry becomes much hazier.

<u>1.3. Relation to tame homotopy</u>. In view of [D] and [DK], one may associate to a pair of tame spaces (S,T) a function complex in the category of simplicial Lazard algebras. This function complex is homotopy equivalent (as a simplicial set) with the pointed mapping space T^S. When T is not tame, however, it is not obvious how one would obtain information about T^S through this technique. The desire to handle the non-tame case motivated this paper. Instead of requiring spaces to be tame, we require them to be R-local, and we restrict the dimensions where their cells may occur.

(The referee has proposed that Dwyer's functor may be able to be specialized suitably to the category of r-connected simplicial sets generated in dimension $\leq m$. This specialization, call it S, might yield information about T^S when S belongs to CW_r^m. To accomplish this, one would attempt to use S in largely the same way that we have used L in this paper.)

<u>1.4. Notations</u>. We work over a fixed subring R of the rationals, and we denote by p the least non-inverted prime, i.e., $p = \inf\{n \in \mathbb{Z}_+ \mid n^{-1} \notin R\}$. In general, then, $\mathbb{Z}[(p-1)!]^{-1} \subseteq R \subseteq \mathbb{Q}$. As in tame homotopy, the relevant dimension ranges vary with a connectivity parameter r, where $r \geq 1$. Following [A1,A2] we introduce several categories.

- SS denotes the category of simplicial sets.
- TOP is the category of pointed topological spaces and pointed continuous maps.
- $CW_r^n(R)$ denotes the full subcategory of TOP, consisting of r-connected R-local CW complexes of dimension $\leq n$. "Dimension" means as an R-local cell complex, e.g., the local n-sphere belongs to $ObCW_r^n(R)$ even though it has topological dimension $n + 1$.
- $HoCW_r^n(R)$ is the category obtained from $CW_r^n(R)$ by collapsing (pointed) homotopy classes of maps.
- DGL(R) is the category of connected dgL's over R. A dgL is <u>free</u> if it is free as a Lie algebra (ignoring the differential); in this case we write it as $(\mathbb{L}(V), \delta)$, where the <u>R-module of</u>

<u>generators</u> $V = \oplus_{i=1}^{\infty} V_i$ is free and positively graded, and the differential δ has degree -1.

- $DGL_r^m(R)$ denotes the full subcategory of $DGL(R)$ whose objects have the form $(\mathbb{L}(V),\delta)$ where $V = \oplus_{i=r}^{m} V_i$, i.e., they are free with all generators occurring in dimensions r through m, inclusive.
- $\mathcal{L}$ denotes the model, introduced in [A1], which carries $CW_r^{m+1}(R)$ to $DGL_r^m(R)$ when $m < rp$.

<u>1.5. Distinguished morphisms in $DGL_r^m(R)$</u>. The category $DGL_r^m(R)$ cannot be made into a closed model category, but we will find it convenient to distinguish three classes of morphisms anyway. Call $f \in Mor\,DGL_r^m(R)$ a <u>weak equivalence</u> if it induces an isomorphism on homology of universal enveloping algebras. It is a <u>cofibration</u> if it splits as an inclusion of free Lie algebras (ignoring the differential), and it is a <u>fibration</u> if it is surjective in dimensions above r. <u>Trivial fibrations</u> are simultaneously fibrations and weak equivalences.

<u>2. Function Complexes in $DGL_r^m(R)$</u>

We will now investigate the possibility of doing homotopy theory in $DGL_r^m(R)$. The dimension limitation, viz., the "m" in $DGL_r^m(R)$, spoils our hope of doing so in the sense of Quillen [Q] or even Baues [B]. We cannot dispense entirely with the bound m, because dgL's exhibit a variety of undesirable behaviors when generator dimensions are permitted to exceed rp. On the other hand, the canonical constructions of turning a map into a fibration or cofibration tend to increase the dimensions of generators, and thus they eventually bump us out of any fixed $DGL_r^m(R)$.

An alternate approach is suggested in [T] and [A1]. We may define for $m < rp$ a homotopy relation on morphisms by utilizing a certain cylinder construction, which raises by one the maximum generator dimension. The gap between m and rp then offers us a "breathing space" in which we can perform the standard constructions approximately $(rp - m)$ times, and thus higher homotopy information is obtainable up to dimension (approximately) $rp - m$. This cylinder construction, known as the <u>Tanré cylinder</u>, is recalled next.

2.1. The Tanré cylinder. This is developed in [T] and [A1] so we provide here only a brief overview. Given a dgL $L = (\mathbb{L}(V), \delta)$ in $DGL_r^m(R)$, where $m < rp$, Tanré associates to it another dgL in $DGL_r^{m+1}(R)$, denoted $IL = (I\mathbb{L}(V), I\delta)$. Taking the set of weak equivalences to be as in 1.5, the dgL IL is a valid cylinder object on L in the sense of [Q] or [B]. In particular, I comes with natural weak equivalences $j_0, j_1\colon id \to I$, and if $L \underset{g}{\overset{f}{\rightrightarrows}} M$ are two morphisms in $DGL_r^m(R)$, then f and g are homotopic if and only if $f \amalg g$ factors through IL. Collapsing homotopy classes gives us a category which we denote by $\mathit{Ho}DGL_r^m(R)$.

We remark that I is not a functor, although $If\colon IL \to IM$ exists non-canonically for each $f\colon L \to M$ in $\mathit{Mor}DGL_r^m(R)$. However, I does satisfy the weak naturality condition $If \circ j_0(L) = j_0(M) \circ f$, $If \circ j_1(L) = j_1(M) \circ f$.

2.2. Constructing the cosimplicial resolution. We construct next an initial segment of a cosimplicial resolution for objects in $DGL_r^m(R)$. We shall use it to define a function complex between two such dgL's. We follow as closely as possible the standard procedure, due to Dwyer and Kan [DK], for constructing cosimplicial resolutions in any closed model category. By a <u>cosimplicial resolution</u> for an object A we mean a (not necessarily functorial) diagram

(1) $$A \rightleftarrows \underset{\sim}{\Delta^1 A} \; \overset{\rightarrow}{\underset{\rightarrow}{\leftleftarrows}} \; \underset{\sim}{\Delta^2 A} \; \ldots \; \underset{\sim}{\Delta^n A} \; \ldots$$

satisfying the usual cosimplicial identities. In (1), each arrow is a weak equivalence; the coface maps are cofibrations, while the codegeneracies are fibrations. (See [DK, Section 4.3] for a precise definition.)

Let us review the Dwyer-Kan construction for a closed model category $\mathcal{C}$. Given an object A, a <u>cylinder</u> on A is an object IA which provides the first stage of a cosimplicial resolution for A. That is, IA fits into a diagram

(2) $$A \underset{i_1}{\overset{i_0}{\rightrightarrows}} A \amalg A \xrightarrow{c} IA \xrightarrow{q} A$$

such that c is a cofibration, q is a trivial fibration, and both composites are the identity on A. This $I(\ )$ need not be a functor, but we do assume the compatibility of $j_0 = ci_0$ and $j_1 = ci_1$ with any If. Typically I arises by factoring the