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Astérisque, tome 191 (1990), p. 211-220
<http://www.numdam.org/item?id=AST_1990__191__211_0>

COHOMOLOGICAL p–NILPOTENCE CRITERIA FOR COMPACT LIE GROUPS

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Introduction

In [Q1] Quillen discussed cohomological criteria for p–nilpotence of finite groups. He proved that for odd primes p a finite group G is p–nilpotent if and only if the restriction map from the mod p cohomology $H^*(G;\mathbb{F}_p)$ to the mod p cohomology $H^*(G_p;\mathbb{F}_p)$ of a p–Sylow subgroup G_p is an F–isomorphism. Recall that a map $A \xrightarrow{\varphi} B$ of graded $\mathbb{F}_p$ algebras is called an F–isomorphism if and only if $a \in \mathrm{Kern}\varphi$ implies $a^n = 0$ for some n and for each $b \in B$ some power b^{p^n} is in the image of φ [Q2]. Furthermore Quillen sketched a proof of the following result which he attributed to Atiyah: If p is any prime and $H^i(G;\mathbb{F}_p) \to H^i(G_p;\mathbb{F}_p)$ is an isomorphism for all sufficiently large i, then G is p–nilpotent.

Quillen's main result in [Q2] can be interpreted as follows: For a compact Lie group G with classifying space BG the F–isomorphism type of $H^*(BG;\mathbb{F}_p)$ is determined by the sets $\mathrm{Rep}(V,G)$ of G–conjugacy classes of homomorphisms from elementary abelian p–groups V to G [HLS]. In particular, one can rephrase Quillen's p–nilpotence criterion in the following form: For an odd prime p a finite group G is p–nilpotent if and only if inclusion induces a bijection $\mathrm{Rep}(V,G_p) \xrightarrow{i} \mathrm{Rep}(V,G)$ for all elementary abelian p–groups V ([HLS; Prop. 4.2.3.]).

If G is a compact Lie group with maximal torus T, normalizer NT, Weyl group $W(G) = NT/T$, then G_p will denote the preimage of W_p in NT. In this case G_p will be called a p–Sylow normalizer and is known to be a good substitute for a p–Sylow subgroup.

S.M.F.
Astérisque 191 (1990)

In this paper we give for odd primes a characterization of those compact Lie groups G for which $\mathrm{Rep}(V, G_p) \to \mathrm{Rep}(V, G)$ is a bijection for all V, or equivalently $H^*(BG; \mathbb{F}_p) \to H^*(BG_p; \mathbb{F}_p)$ is an F–isomorphism (Theorem 2.1.). The possibility of such a characterization was already mentioned in [HLS, Sect. 4.2.5.]. It seems appropriate to call such groups p–nilpotent compact Lie groups. We will also generalize Atiyah's criterion to the compact Lie group case (Theorem 2.5.). Our interest in such characterizations comes from the importance of BG_p for the study of the (stable) homotopy type of BG.

The paper is organized as follows. In section 1 we give the precise definition of a p–nilpotent compact Lie group and discuss some properties of such groups. We do not intend a systematic group theoretical study of this concept but will rather concentrate on properties which are relevant for our cohomological characterizations. These characterizations are stated and proved in section 2.

The author would like to thank L. Evens and L. Schwartz for helpful discussions on this subject. This work was started while the author stayed at Northwestern University. He is grateful to the DFG and Northwestern University for supporting this stay and to the people at Northwestern for providing a pleasant and stimulating atmosphere.

1. p–nilpotent compact Lie groups

1.1 DEFINITION. *A compact Lie group G is called p–nilpotent if and only if there is a finite normal subgroup N of order prime to p which together with G_p generates G.*

1.2 REMARKS.

(a) For finite groups this reduces to the classical definition of p–nilpotence. Then N consists of all elements of order prime to p and G/N is isomorphic to G_p, i.e. G is a semidirect product $N \rtimes G_p$. In this case N is also called the normal p complement of G_p in G.

(b) In the compact Lie group case G is in general not a semidirect product. For example, if $G = \langle S^1,\ x, y \mid [x, S^1] = [y, S^1] = x^3 = y^3 = 1,$ $[x, y] = \zeta$ with ζ a primitive 3rd root of unity in $S^1\rangle$ and $p \neq 3$, then

$G_p = S^1$ and the normal subgroup $N = \langle x, y\rangle$ shows that G is p–nilpotent. However, $N \cap G_p \neq \{1\}$ and hence $G \not\cong N \rtimes G_p$. It is also obvious that G is not a semidirect product $\widetilde{N} \rtimes G_p$ for some other $\widetilde{N} \lhd G$.

Our definition of p–nilpotence above will be justified by the results below, which together with this example show that it would not be adequate to require the existence of a finite normal p–complement in the compact Lie group case.

1.3 PROPOSITION. *Let G be a compact Lie group and p be any prime. Then the following statements are equivalent.*

(a) *G is p–nilpotent.*

(b) $\mathrm{Rep}(Q, G_p) \xrightarrow{i} \mathrm{Rep}(Q, G)$ *is a bijection for all p–groups Q.*

(c) *If Q is any finite p–subgroup of G, then $N_G(Q)/C_G(Q)$, the quotient of the normalizer of Q in G by the centralizer of Q in G, is a finite p–group.*

(d) *Each finite subgroup H of G is p–nilpotent.*

(e) *G is a finite extension of a torus, i.e. there exists an exact sequence $T \hookrightarrow G \twoheadrightarrow \pi$ with π finite, and G has a finite p-nilpotent subgroup H with $H/H \cap T = \pi$ and $T_p = \{t \in T \mid t^p = 1\} \subset H$.*

(f) *G is an extension of a torus by a finite p–nilpotent group π and the conjugation action of the normal p–complement ν of π_p in π is trivial on T.*

Proof. (a) $\Rightarrow$ (b): Onto is equivalent to saying that any p–subgroup Q of G is conjugate to a subgroup of G_p, i.e. that the Q–set G/G_p has a nonempty Q–fixed point set $(G/G_p)^Q$. This follows from $\chi((G/G_p)^Q) \equiv \chi(G/G_p) \not\equiv 0 \bmod p$ where χ denotes Euler characteristic (cf. [HLS; Prop. 4.2.1.]).

To show that i is $1-1$ consider the projection $G_p \xrightarrow{\pi} G_p/G_p \cap N \cong G/N$. It suffices to show that π induces an injection on $\mathrm{Rep}(Q, ?)$. So let α_1, α_2 be two homomorphisms with $\pi\alpha_1 = g\pi\alpha_2 g^{-1}$ for some $g \in G_p$. By factoring out the kernel we may assume that $\pi\alpha_1$ is mono. Identify Q with its image in $G_p/G_p \cap N$. Then α_1 and $g\alpha_2 g^{-1}$ are sections of $\pi^{-1}(Q) \xrightarrow{\pi} Q$. Now $\mathrm{Kern}\,\pi = G_p \cap N$ is a subgroup of T of order prime to p and hence

$H^1(Q, G_p \cap N) = 0$, i.e. α_1 and $g\alpha_2 g^{-1}$ are even conjugate by an element in $G_p \cap N$ and we are done.

(b) $\Rightarrow$ (c): For any group G the automorphism group $\mathrm{Aut}(Q)$ acts on $\mathrm{Rep}(Q,G)$. If Q is a subgroup of G, then $N_G(Q)/C_G(Q)$ identifies naturally with the isotropy subgroup of the inclusion $Q \hookrightarrow G$, considered as an element in the $\mathrm{Aut}(Q)$–set $\mathrm{Rep}(Q,G)$.

Now (b) implies that we can assume that Q is a subgroup of G_p and that it suffices to show that $N_{G_p}(Q)/C_{G_p}(Q)$ is a p–group. So suppose that $x \in N_{G_p}(Q)$ has order prime to p in $N_{G_p}(Q)/C_{G_p}(Q)$. As in [HLS, sect. 4.3.] we may assume that x itself has order prime to p, i.e. $x \in T$. Then one sees as in [HLS, Lemma 4.3.3.] that x acts trivially on the quotient of Q by its Frattini–subgroup $\phi(Q)$ and hence trivially on Q (cf. [H, Satz III 3.18.]). Therefore x is in $C_{G_p}(Q)$ and we are done.

(c) $\Rightarrow$ (d): If Q is a subgroup of H, then $N_H(Q)/C_H(Q)$ is a subgroup of $N_G(Q)/C_G(Q)$ and hence the Frobenius criterion [H, Satz IV, 5.8.] implies that H is p–nilpotent.

For the remaining implications we need a Lemma. For a natural number ℓ let T_ℓ denote $\{t \in T \mid t^\ell = 1\}$.

1.4 LEMMA. *Let G be an extension of a torus T by a finite group π of order $|\pi|$. Then there is a finite subgroup F of G with $F/F \cap T = \pi$ and $F \cap T = T_{|\pi|}$.*

Proof. Interpret the (class of the) extension $T \hookrightarrow G \twoheadrightarrow \pi$ as an element $[e] \in H^2(\pi; T)$ and use that $|\pi| \cdot [e] = 0$ together with the long exact cohomology sequence arising from the short exact sequence $T_{|\pi|} \hookrightarrow T \xrightarrow{\bullet|\pi|} T$ of π–modules.

□

We continue with the proof of Proposition 1.3.

(d) $\Rightarrow$ (e): Assume that G is not a finite torus extension. Then $G_{(1)}$, the connected component of 1, is not abelian and hence contains a compact connected nonabelian Lie group of rank 1, i.e. either $SO(3)$ or $SU(2)$. Now $SO(3)$ contains A_4, the alternating group on four letters, as symmetry group