

Astérisque

MARTIN MARKL

The rigidity of Poincaré duality algebras and classification of homotopy types of manifolds

Astérisque, tome 191 (1990), p. 221-237

<http://www.numdam.org/item?id=AST_1990__191__221_0>

© Société mathématique de France, 1990, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

THE RIGIDITY OF POINCARÉ DUALITY ALGEBRAS AND CLASSIFICATION OF HOMOTOPY TYPES OF MANIFOLDS

MARTIN MARKL

INTRODUCTION

This paper is devoted to the study of homotopy types of simply connected rational Poincaré duality spaces. We will frequently use the language and results of rational homotopy theory, a good common reference is the book [14].

So, let X be a rational Poincaré duality space of the (top) dimension n , i.e. a simply connected space, whose rational cohomology algebra is a Poincaré duality algebra of the formal dimension n ; see §3. It is well-known (see also §3) that X has the rational homotopy type of a space of the form $Y \cup_h e^n$, where Y is a simply connected CW -complex of dimension $< n$ and $h : S^{n-1} = \partial e^n \rightarrow Y$ is a continuous map. The space Y , defined uniquely up to rational homotopy type, will be called (with some inaccuracy) the skeleton of X and will be denoted by $X_{<n}$. If X is a simply connected n -dimensional manifold, the construction above can be described even more geometrically: take $X \setminus B^n$, where B^n is a (sufficiently small) n -dimensional open disc. It is easy to remark that the n -dimensional manifold with boundary, $X \setminus B^n$, has the same rational homotopy type as the skeleton $X_{<n}$, constructed above.

Recall that two simply connected spaces X and Y are said to have the same \mathbf{k} -homotopy type, where \mathbf{k} is a field of characteristic zero, if their Quillen minimal models [14; III.3.(1)] are isomorphic over \mathbf{k} ; this fact will be denoted by $X \sim_{\mathbf{k}} Y$. Of course, for $\mathbf{k} = \mathbf{Q}$ we get the usual definition of the rational homotopy equivalence.

Fix an n -dimensional rational Poincaré duality space X (simply connected by definition). The aim of this paper is to give a description of the set $PDS_{\mathbf{k}}(X)$ of all \mathbf{k} -homotopy types of rational Poincaré duality spaces Y whose skeleta $Y_{<n}$ have the same rational homotopy type as the skeleton $X_{<n}$ of X , when X is formal. It is interesting to point out that the set $PDS_{\mathbf{k}}(X)$ is, according to rational surgery results [3],

for $n \not\equiv 0 \pmod{4}$ naturally isomorphic to the set $Man_{\mathbf{k}}(X)$ of all \mathbf{k} -homotopy types of n -dimensional compact simply connected manifolds M with $M_{<n} \sim_{\mathbf{Q}} X_{<n}$.

The first attempt towards the description of $PDS_{\mathbf{k}}(X)$ was made in [12], where it is stated [12; Theorem 1] that the rational homotopy type of a rational Poincaré duality space is uniquely determined by the rational homotopy type of its skeleton, if the cohomology algebra of X is fixed. Here we will always suppose that X is formal, the hypothesis taken by M. Aubry [1,2].

We give here a complete description of the set $PDS_{\mathbf{k}}(X)$ in terms of usual algebraic objects – Galois cohomology and induced maps – when X is formal. Using this description, we are able to prove, for example, that the \mathbf{k} -homotopy type of a rational Poincaré duality space is uniquely determined by its skeleton provided that \mathbf{k} is algebraically closed. We prove also that the set $PDS_{\mathbf{k}}(X)$ (and hence also $Man_{\mathbf{k}}(X)$) is finite for fields satisfying $[\bar{\mathbf{k}} : \mathbf{k}] < \infty$ (for example for $\mathbf{k} = \mathbf{R}$, the case of real homotopy types). As an example of explicit calculations we construct a large class of Poincaré duality spaces X for which the set $PDS_{\mathbf{k}}(X)$ consists of the \mathbf{k} -homotopy type of X only, \mathbf{k} arbitrary. On the other hand, we give an example of a manifold M , for which the set $PDS_{\mathbf{Q}}(M)$ is infinite.

The algebraic counterpart of the description of $PDS_{\mathbf{k}}(X)$ is the following classification problem: let H^* be a Poincaré duality algebra of formal dimension n , how to describe the set $PDA_{\mathbf{k}}(H^*)$ of all isomorphism classes of Poincaré duality algebras H'^* with $H'^*/H'^n \cong H^*/H^n$. Our approach to the study of the set $PDA_{\mathbf{k}}(H^*)$ is based on a rigidity property of Poincaré duality algebras over an algebraically closed field and on the usual method of descent. We hope that this approach can be used even in more general situation – for the classification of all Gorenstein rings R having the “skeleton” $R/Socle(R)$ fixed (see [15]).

Our paper is organized as follows. In the first paragraph we prove a rigidity theorem for Poincaré duality algebras. The proof of this statement is based on a deliberate use of the deformation theory; note that this machinery has already been systematically used in rational homotopy theory in [4]. As a by-product we obtain a characterization of Poincaré duality in terms of Harrison cohomology. These results are in the next paragraph applied to the solution of our classification problem for Poincaré duality algebras. The main result of this section is Theorem 2.7. In the third paragraph the algebraic theory is applied to the study of the set $PDS_{\mathbf{k}}(X)$ as introduced above, a

description is given in Theorem 3.2. Notice that both Theorem 3.2 and the forthcoming examples explicitly describe the effect of the ground field \mathbf{k} on the structure of $PDS_{\mathbf{k}}(X)$, hence all the material of this paragraph can be considered as a contribution to the study of descent and non-descent phenomena in rational homotopy theory in the spirit of [10].

I would like to express here my thanks to Ștefan Papadima for drawing my attention to the possible use of descent methods. Also the formulation of the condition iii) of Theorem 1.5 is due to him. I wish also to acknowledge my indebtedness to the referee for useful comments and references.

1. RIGIDITY OF POINCARÉ DUALITY ALGEBRAS

As usually, by a Poincaré duality algebra (over a field \mathbf{k}) of the formal dimension n is meant a (finite dimensional) graded commutative \mathbf{k} -algebra $H^* = \bigoplus_{i \geq 0} H^i$ such that H^n is isomorphic to \mathbf{k} , $H^i = 0$ for $i > n$ and the bilinear form $B : H^* \otimes H^* \rightarrow \mathbf{k}$ of degree $-n$ defined by

$$B(x, y) = \begin{cases} x \cdot y \in \mathbf{k} \cong H^n & \text{for } \deg(x) + \deg(y) = n \\ 0 & \text{otherwise} \end{cases}$$

is nondegenerate in the usual sense. All Poincaré duality algebras (and Poincaré duality spaces) in this paper are supposed to have the same formal dimension equal to a given natural number n .

1.1. For a graded commutative algebra A^* denote:

$$\begin{aligned} \mathcal{B}(A^*) &= \left\{ \begin{array}{l} \text{all bilinear forms } B : A^* \otimes A^* \rightarrow \mathbf{k} \text{ of degree } -n \text{ such} \\ \text{that } B(x, y) = (-1)^{\deg(x)\deg(y)} B(y, x) \text{ for } x, y \in A^* \end{array} \right\}, \\ \mathcal{M}(A^*) &= \{ B \in \mathcal{B}(A^*); B(xy, z) = B(x, yz) \text{ for } x, y, z \in A^* \}, \\ \mathcal{P}(A^*) &= \{ B \in \mathcal{M}(A^*); B \text{ is nondegenerate on } A^{>0} \otimes A^{>0} \} \quad \text{and} \\ G(A^*) &= \text{Aut}(A^*) = \text{the group of graded automorphisms of } A^*. \end{aligned}$$

Notice that all the sets above have the natural structure of a (not necessarily irreducible) algebraic variety. The geometry of $\mathcal{M}(A^*)$ is extremely simple—as all the defining equations are linear, it is isomorphic to an affine space. The set $\mathcal{P}(A^*)$ is

plainly Zariski-open and dense in $\mathcal{M}(A^*)$. The group $G(A^*)$ acts naturally from the left on $\mathcal{B}(A^*)$ by

$$\phi(B)(x, y) = B(\phi^{-1}(x), \phi^{-1}(y)).$$

Clearly $G(A^*)\mathcal{M}(A^*) \subset \mathcal{M}(A^*)$ and $G(A^*)\mathcal{P}(A^*) \subset \mathcal{P}(A^*)$. The action of $G(A^*)$ is plainly continuous in the Zariski topology.

We call an algebra A^* a fragment, if it is of the form

$$A^* = H_{<n}^* := H^*/H^n$$

for a Poincaré duality algebra H^* . The algebra $H_{<n}^*$ will be called the *skeleton* of H^* . Here $H_{<n}^*$ is defined as a quotient, but after having chosen a section, we may as well consider it as a subset of H^* .

It is interesting to remark that it is always possible to decide in finitely many steps whether a given graded commutative algebra A^* is a fragment or not. To this end, find at first a basis of the affine space $\mathcal{M}(A^*)$. Our algebra A^* is then a fragment if and only if the polynomial function, representing the determinant, is not equal to zero on $\mathcal{M}(A^*)$ identically.

This characterization problem for fragments is the special case of the problem of deciding when a given local ring is a factor of a Gorenstein ring by the socle, see [15].

1.2. For a fragment A^* consider the set $\tilde{\mathcal{M}}(A^*)$ of all graded commutative algebras H^* with $H^i = 0$ for $i > n$, $H^n \cong \mathbf{k}$ and $H_{<n}^*$ isomorphic to A^* . For $H^* \in \tilde{\mathcal{M}}(A^*)$ choose an isomorphism $r : H^n \rightarrow \mathbf{k}$ and define $B \in \mathcal{M}(A^*)$ by $B(x, y) = r(x \cdot y) \in \mathbf{k}$. The form B is defined canonically up to a nonzero multiple from \mathbf{k} . Keeping in mind this ambiguity, we can write $H^* = (A^*, B)$. Notice that H^* is a Poincaré duality algebra if and only if $B \in \mathcal{P}(A^*)$.

1.3. Let $A^* = H_{<n}^*$ be a fragment and denote by $PDA_{\mathbf{k}}(H^*)$ the set of all isomorphism classes of Poincaré duality \mathbf{k} -algebras having the skeleton isomorphic to A^* . We claim that the presentation 1.2 induces a bijection between $PDA_{\mathbf{k}}(H^*)$ and the orbit space $\mathcal{P}(A^*)/G(A^*)$ provided that \mathbf{k} algebraically closed.

To verify this, notice at first that each algebra from $PDA_{\mathbf{k}}(H^*)$ is isomorphic to an algebra H'^* with $H'^*_{<n} = A^*$. Hence we can suppose immediately that $H'^*_{<n} = A^*$ for each $H'^* \in PDA_{\mathbf{k}}(H^*)$. Let $H'^* = (A^*, B')$ and $H''^* = (A^*, B'')$ be two algebras from