J. P. May

Some remarks on equivariant bundles and classifying spaces

Astérisque, tome 191 (1990), p. 239-253
<http://www.numdam.org/item?id=AST_1990__191__239_0>

Some remarks on equivariant bundles and classifying spaces

by J. P. May

Let Π be a normal subgroup of a topological group Γ with quotient group G; subgroups are understood to be closed. A principal $(\Pi;\Gamma)$-bundle is the projection to orbits $E \to E/\Pi$ of a Π-free Γ-space E. (Function spaces excepted, our Γ-spaces are to be of the homotopy type of Γ-CW complexes, and similarly for other groups.) For a G-space X, let $\mathcal{B}_G(\Pi;\Gamma)(X)$ denote the set of equivalence classes of principal $(\Pi;\Gamma)$-bundles over X. For a space X, let $\mathcal{B}(\Pi)(X)$ denote the set of equivalence classes of principal Π-bundles over X. Let X_G denote the Borel construction $EG \times_G X$ associated to a G-space X. We write

$$\mathcal{B}_G(\Pi;\Gamma)(EG \times X) = \mathcal{B}(\Pi;\Gamma)(X_G)$$

to emphasize that this set depends only on X_G as a space over BG. Equivalently, $\mathcal{B}(\Pi;\Gamma)(X_G)$ is the set of equivalence classes of free Γ-spaces P with a given equivalence $P/\Pi \cong EG \times X$ of G-bundles over $P/\Gamma \cong X_G$. We shall see that the calculation of this set reduces to a nonequivariant lifting problem, and we think of it as essentially a problem in ordinary nonequivariant bundle theory. In fact, in the classical case $\Gamma = G \times \Pi$, passage from P to P/G specifies a natural bijection

$$\Theta\colon \mathcal{B}(\Pi;G \times \Pi)(X_G) \to \mathcal{B}(\Pi)(X_G).$$

The projection $EG \times X \to X$ induces a natural map

$$\Psi\colon \mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G).$$

S.M.F.
Astérisque 191 (1990)

In the classical case, $\Phi = \Theta\Psi$ is just the Borel construction on bundles. One of our goals is to determine how near the passage Ψ from equivariant bundle theory to ordinary bundle theory is to being an isomorphism. For example, we shall obtain the following result, which is essentially just an exercise in covering space theory.

THEOREM 1. *If* Γ *is discrete, then* $\Psi\colon \mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G)$ *is a bijection for any* G*-space* X*. If* Π *(but not necessarily* G*) is discrete, then* $\Phi\colon \mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ *is a bijection for any* G*-space* X*.*

We shall see that the following deeper result is a consequence of the Sullivan conjecture. The phrase "(strong) mod p equivalence" will be explained in due course.

THEOREM 2. *Let* G *be an extension of a torus by a finite* p*-group. If* Γ *is a compact Lie group, then the natural transformation* $\Psi\colon \mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G)$ *is represented by a mod p equivalence of classifying* G*-spaces. Therefore, if* Π *is a compact Lie group, then* $\Phi\colon \mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ *is represented by a mod p equivalence of classifying* G*-spaces. If* G *is a finite* p*-group, then the transformations* Ψ *and* Φ *are represented by strong mod p equivalences of classifying* G*-spaces.*

Restricting Π instead of G, we obtain the following theorem, which is the main result of [7].

THEOREM 3. *If* G *and* Π *are compact Lie groups with* Π *Abelian, then* $\Phi\colon \mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ *is a bijection for any* G*-space* X*.*

In a preprint version of this paper, the following assertion was claimed as a theorem.

ASSERTION 4. *Under the hypotheses of theorem 3, there is also a natural bijection*

$$\mathcal{B}G(\Pi;G \times \Pi)(X) \cong \mathcal{B}(\Pi)(X/G) \times Nat(\pi_0(X), R_\pi).$$

The fact that this assertion is false was discovered by John Wicks, a student at Chicago, who showed that, with $\Pi = S^1$ and $G = Z_2$, it implies an incorrect calculation of characteristic classes. Since the nature of the assertion and the mistake in its proof may be of interest, we shall discuss these matters in an Appendix.

The three theorems above are direct interpretations of results about equivariant classifying spaces, namely Theorems 5, 9, and 10 below. There is a universal example $E(\Pi;\Gamma) \to B(\Pi;\Gamma)$ of a principal $(\Pi;\Gamma)$-bundle. Up to Γ-homotopy type, the Γ-space $E(\Pi;\Gamma)$ is characterized by the requirement that, for $\Omega \subset \Gamma$, the fixed point space $E(\Pi;\Gamma)^\Omega$ be contractible if $\Omega \cap \Pi = e$ and empty otherwise.

By universality, we have a natural bijection

$$(*) \qquad \mathcal{B}G(\Pi;\Gamma)(X) \cong [X, B(\Pi;\Gamma)]_G,$$

where homotopy classes of unbased G-maps are understood. In particular, we have natural bijections

$$\mathcal{B}G(\Pi;\Gamma)(EG \times X) \cong [EG \times X, B(\Pi;\Gamma)]_G \cong [X, Map(EG, B(\Pi;\Gamma))]_G.$$

Let $p: X_G \to BG$ be the evident bundle and let $q: \Gamma \to G$ be the quotient homomorphism. Let $[X_G, B\Gamma]/BG$ be the set of homotopy classes of maps $f: X_G \to B\Gamma$ such that $Bq \circ f = p$ and define

$\mathrm{Sec}(EG, B\Gamma)$ to be the G-space of maps $\varphi: EG \to B\Gamma$ such that $Bq \circ \varphi = p: EG \to BG$. A central idea in this paper is the modelling of classifying spaces by such spaces of sections. We introduce this idea by observing that the previous bijections are equivalent to

$$(\#) \qquad \mathcal{B}(\Pi;\Gamma)(X_G) \cong [X_G, B\Gamma]/BG \cong [X, \mathrm{Sec}(EG, B\Gamma)]_G.$$

This should be clear from the equivalent bundle theoretic descriptions of the left sides already given, but we want to see it directly on the classifying space level. Since $E\Gamma$ is Π-free, the universal property of $E(\Pi;\Gamma)$ gives a Γ-map $\nu: E\Gamma \to E(\Pi;\Gamma)$, unique up to Γ-homotopy. The Γ-map $(Eq,\nu): E\Gamma \to EG \times E(\Pi;\Gamma)$ is clearly a Γ-homotopy equivalence, where Γ acts through q on EG, and it is a fiber Γ-homotopy equivalence provided we choose a model for $E\Gamma$ such that $Eq: E\Gamma \to EG$ is a Γ-fibration. Passing to orbits over Γ by first passing to orbits over Π and then over G, we obtain a homotopy equivalence

$$B\Gamma \to EG \times_G B(\Pi;\Gamma) = B(\Pi;\Gamma)_G$$

over BG. (Lemma 11 at the end will generalize this equivalence.) We have an evident G-homeomorphism $\mathrm{Sec}(EG, X_G) \cong \mathrm{Map}(EG, X)$ for any G-space X, and there results a G-homotopy equivalence

$$\xi: \mathrm{Sec}(EG, B\Gamma) \to \mathrm{Sec}(EG, B(\Pi;\Gamma)_G) \cong \mathrm{Map}(EG, B(\Pi;\Gamma)).$$

Via the projection $EG \to pt$ and use of a chosen homotopy inverse to ξ, we obtain a G-map

$$\alpha: B(\Pi;\Gamma) \to \mathrm{Sec}(EG, B\Gamma)$$

which induces the transformation Ψ under the isomorphisms $(*)$ and $(\#)$. In order to prove Theorem 1, we model $E(\Pi;\Gamma)$ as a space of