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MAPS BETWEEN p–COMPLETIONS OF THE CLARK–EWING SPACES X(W,p,n)

by

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Abstract. Let Z_p denote the ring of p–adic integers. Let $W \subset GL(n,Z_p)$ be a finite group such that p does not divide the order of W. The group W acts on $K((Z_p)^n,2)$. Let $X(W,p,n)_p$ be the p–completion of the space $K((Z_p)^n,2) \underset{W}{\times} EW$. We classified homotopy classes of maps between spaces $X(W,p,n)_p$.

0. INTRODUCTION

Let Z_p denote the ring of p–adic integers. Let Y_p denote the p–completion of a space Y.

Let T be a torus and let $W \subset GL(\pi_1(T) \otimes Z_p)$ be a fintie group. The group W acts on the space $(BT)_p$. Let

$$X(W,p,T) := ((BT)_p \times_W EW)_p$$

where EW is a contractible space equipped with a free action of W.

The aim of this paper is to apply the program from [1] to study maps between spaces X(W,p,T). The starting point was an attempt to generalize one result of Hubbuck (see [8] Theorem 1.1.). The plan of work will follow closely that of [3] and [13].

Example. *Let* G *be a connected, compact Lie group,* T *its maximal torus and* W *its Weyl group. If* p *does not divide the order of* W *then* $(BG)_p \approx (BT \times_W EW)_p$.

This example suggests the following defintion.

Definition. *Let us set* $X = X(W,p,T)$. *We shall call* T *a maximal torus of* X *and* W *a Weyl group of* X.

The projection $(BT)_p \times EW \longrightarrow (BT)_p \times_W EW$ *induces a map* $i : BT \longrightarrow X$. *We shall call* $i : BT \longrightarrow X$ *a structure map of* X.

We point out that in [5] A. Clark and J. Ewing studied cohomology algebras of spaces $(BT)_p \times_W EW$. We warn the reader that our notation is different from the notation used in [5]. The space $X(W,p,T)$ is the p–completion of the Clark–Ewing space $X(W,p,\text{rank } T)$.

Through the whole paper we shall assume that p is an odd prime. We need this assumption to show Proposition 1.1. It is clear that this assumption is not essential, however we were not able to overcome technical difficulties for $p = 2$.

Now we shall state our main results.

Let us set $X = X(W,p,T)$ and $X' = X(W',p,T')$.

THEOREM 1. Assume that p *does not divide the orders of* W *and* W'. *Then for any map* $f : X \longrightarrow X'$ *there is a map* $\Upsilon : (BT)_p \longrightarrow (BT')_p$ *such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ (BT)_p & \xrightarrow{\quad \Upsilon \quad} & (BT')_p \end{array}$$

commutes up to homotopy. Moreover we have:

a) *if* $\Upsilon' : (BT)_p \longrightarrow (BT')_p$ *is such that* $f \circ i$ *is homotopic to* $i' \circ \Upsilon'$ *then there is* $w \in W'$ such that $w \circ \Upsilon'$ *is homotopic to* Υ,
b) *for any* $w \in W$ *there is* $w' \in W'$ *such that* $\Upsilon \circ w$ *is homotopic to* $w' \circ \Upsilon$.

The group W acts on $\pi_1(T) \otimes Z_p$, hence W acts on $\pi_1(T) \otimes R$ for any Z_p–module R.

DEFINITION 1. Let R *be a* Z_p*–algebra. We say that a homomorphism of* R*–modules*

$$\varphi : \pi_1(T) \otimes R \longrightarrow \pi_1(T') \otimes R$$

is admissible if for any $w \in W$ *there is* $w' \in W'$ *such that* $\varphi \circ w = w' \circ \varphi$.
We say that two admissible maps φ *and* ψ *from* $\pi_1(T) \otimes R$ *to* $\pi_1(T') \otimes R$ *are equivalent if there is* $w \in W'$ *such that* $w \circ \varphi = \psi$.

It is clear that the relation defined above is an equivalence relation on the set of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$. We shall denote by $\mathrm{Ahom}_R(T,T')$ the set of equivalence classes of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$.

Let us notice that the map $\pi_1(\Upsilon)$ induced by Υ from Theorem 1 on fundamental groups is admissible for $R = Z_p$. This map is unique up to the action of W', so any map $f : X \longrightarrow X'$ determines uniquely an equivalence class of $\pi_1(\Upsilon)$ in $\mathrm{Ahom}_{Z_p}(T,T')$ which we shall denote by $\chi(f)$.

THEOREM 2. Let us assume that p *does not divide the orders of* W *and* W′*. Then the natural map*

$$\chi : [X,X'] \longrightarrow \mathrm{Ahom}_{Z_p}(T,T')$$

is bijective.

For any space Y we set

$$H^*(Y,\mathbb{Q}_p) := H^*(Y,Z_p) \otimes \mathbb{Q} ,$$

where $\mathbb{Q}_p$ is a field of p–adic numbers.

THEOREM 3. Let us assume that p *does not divide the orders of* W *and* W′*. Then the natural map*

$$\phi : [X,X'] \longrightarrow \mathrm{Hom}(H^*(X',\mathbb{Q}_p), H^*(X,\mathbb{Q}_p))$$

is injective.

We denote by $K^0(\ ,R)$ the 0^{th}–term of complex K–theory with R–coefficients. Let $\mathcal{O}_R$ be the set of operations in $K^0(\ ,R)$. The functor $K^0(\ ,R)$ is equipped with the natural augmentation $K^0(\ ,R) \longrightarrow R$. Let $\mathrm{Hom}_{\mathcal{O}_R}(K^0(X',R),K^0(X,R))$ be the set of R–algebra homomorphisms which commute with the action of $\mathcal{O}_R$ and augmentations.

THEOREM 4. *If* p *does not divides the order of* W *and* W′, *then the natural map*

$$\psi : [X,X'] \longrightarrow \mathrm{Hom}_{\mathcal{O}_{Z_p}}(K^0(X',Z_p),K^0(X,Z_p))$$

is bijective.

We can formulate our results in a nice categorical way.

We shall define a category $Z_p - \mathrm{Rep}$ in the following way. Objects of the category Z_p–Rep are representations $\rho : W \longrightarrow GL(M)$ where M is a free, finitely generated Z_p–module, W is a finite group and p does not divide the order of W. It remains to define morphisms in this category. If $\theta : W \longrightarrow GL(M)$ and $\theta' : W' \longrightarrow GL(M')$ are two objects of $Z_p - \mathrm{Rep}$, we say that a homomorphism of Z_p–modules $f : M \longrightarrow M'$ is admissible if for each $w \in W$ there is $w' \in W'$ such that $f \circ w = w' \circ f$. We say that two admissible homomorphisms f and g from M to M′ are equivalent if there is $w \in W'$ such that $f = w' \circ g$. We shall denote by $\mathrm{Ahom}(\theta,\theta')$ the set of equivalence classes of admissible homomorphisms from M to M′. The set $\mathrm{Ahom}(\theta,\theta')$ is the set of morphisms from θ to θ' in the category $Z_p -$ Rep. The category $Z_p -$ Rep is equipped with the product defined in the following way:

$$(\theta{:}W \longrightarrow GL(M)) \oplus (\theta'{:}W' \longrightarrow GL(M')) = \theta\oplus\theta' : W \times W' \longrightarrow GL(M \oplus M').$$

The product of morphisms is defined in the obvious way.

We denote by Ht(p) the category whose objects are spaces X(W,p,T) such