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DERIVED FUNCTORS OF THE DESTABILIZATION and THE ADAMS SPECTRAL SEQUENCE

by Said ZARATI

Introduction

Let A be the modulo 2 Steenrod algebra, $\mathcal{M}$ the category of graded A-modules and A-linear maps of degree zero, and $\mathcal{U}$ the full sub-category of $\mathcal{M}$ whose objects are unstable A-modules. We denote by D : $\mathcal{M}$ ---> $\mathcal{U}$ the destabilization functor and by D_s, $s \geq 0$, its derived functors. We have a natural transformation : D_s ---> $\Sigma D_s \Sigma^{-1}$, $s \geq 0$, induced by the adjoint of the identity $\Omega D = D \Sigma^{-1}$ where Σ^m, $\mathcal{M}$ ---> $\mathcal{M}$, $m \in \mathbb{Z}$, is the m^{th} suspension functor and Ω is the left adjoint of Σ : $\mathcal{U}$ ----> $\mathcal{U}$.

In this note we prove the following theorem wich will be more precise in section 2.3.

Theorem 1.1. Let M be a nil-closed unstable A-module. Then the natural map $\Omega D_s \Sigma^{-s} M$ ---> $D_s \Sigma^{-s-1} M$ is an isomorphism for every $s \geq 0$.

Using the higher Hopf invariants introduced in [7] we prove the following property of the Adams spectral sequence, in the modulo 2 cohomology, for the group {X,Y} of homotopy classes of stable maps from X to Y, in certain cases.

Theorem 1.2. : Let X and Y two pointed CW-complexes such that

(i) $\bar{H}^*(X,\mathbb{F}_2) \simeq \Sigma^2 I$ where ΣI is an injective unstable A-module.

(ii) $\bar{H}^*(Y;\mathbb{F}_2)$ is gradually finite and nil-closed.

Then, the Adams spectral sequence for the group {X,Y} degenerate at the E_2-term : $E_2^{s,s} \approx E_r^{s,s}$ for every $r \geq 2$ and $s \geq 0$.

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The infinite real projective space $\mathbb{R}P^{\infty}$ is an example of a space Y satisfying the hypotheses of theorem 1.2.

The organization of the rest of this note is as follows. In section 2 we give a characterization of nil-closed A-modules which allows us to prove the theorem 1.1 (see theorem 2.3.3). Section 3 gives the proof of theorem 1.2 and an application. We finish this note by a remark concerning the case p > 2.

All cohomology is taken with $\mathbb{F}_2$ coefficients. We write $H^*(\ )$ for $H^*(\ ;\mathbb{F}_2)$ and we denote by $\bar{H}^*(\ )$ the reduced modulo 2 cohomology.

2. Derived functors of the destabilization

2.1. Let A be the modulo 2 Steenrod algebra. We denote by $\mathcal{M}$ the category whose objects are graded A-modules ($M = \{M^n, \quad n \in \mathbb{Z}\}$) and whose morphisms are A-linear maps of degree zero. We denote by $\mathcal{U}$ the full sub-category of $\mathcal{M}$ whose objects are unstable A-modules (an A-module M is called unstable if $Sq^i x = 0$ for every x in M^n and every $i > n$; in particular $M^n = 0$ if $n < 0$).

The forgetful functor $\mathcal{U} \dashrightarrow \mathcal{M}$ has a left adjoint functor D : $\mathcal{M} \dashrightarrow \mathcal{U}$, called the destabilization functor, which satisfies : $\mathrm{Hom}_{\mathcal{M}}(M,N) = \mathrm{Hom}_{\mathcal{U}}(DM,N)$ for every A-module M and every unstable A-module N. The functor D : $\mathcal{M} \dashrightarrow \mathcal{U}$ is right exact, we denote $D_s : \mathcal{M} \dashrightarrow \mathcal{U}$, $s \geq 0$, its derived functors. One of the motivations for the study of the derived functors of the destabilization is the following isomorphism :

(2.1) $$\mathrm{Ext}^s_{\mathcal{M}}(M,I) \simeq \mathrm{Hom}_{\mathcal{U}}(D_sM,I)$$

for every A-module M and every unstable injective A-module I.

Let $\Sigma^m : \mathcal{M} \dashrightarrow \mathcal{M}$, $m \in \mathbb{Z}$, the m^{th} suspension functor

which associates to a module $M = \{M^n, n \in \mathbb{Z}\}$ the module $\Sigma^m M = \{M^{n-m}, n \in \mathbb{Z}\}$. The A-module structure on $\Sigma^m M$ is given by $Sq^i(\Sigma^m x) = \Sigma^m Sq^i x$, x in M. The computation of $D_s\Sigma^{-t}M$, where M is an unstable A-module, is done by Lannes and Zarati [5] for $t \leq s$.
In this paragraph we will compute $D_s\Sigma^{-(s+1)}M$ for a particular unstable A-modules called nil-closed. First let us recall the definition and some properties of nil-closed unstable A-modules.

2.2. Nil-closed unstable A-modules [1], [6]

Definition 2.2.1 An unstable A-module M is called reduced if the cup-square $Sq^n : M^n \dashrightarrow M^{2n}$, $x \dashrightarrow Sq^n x$, is injective for every $n \geq 0$.

Remark 2.2.2 We can verify easily that an unstable A-module is reduced if and only if it does not contain a non trivial nilpotent sub-A-module. An unstable A-module N is called nilpotent if for every x in M^n, there exist $r \geq 0$ such that $Sq^{2^r n} \ldots\ldots Sq^n x = 0$.

Definition 2.2.3. An unstable A-module M is called nil-closed if
(i) M is reduced
(ii) An element x in M of even degree is in the image of the cup-square if and only if $Q_i x = 0$, for all $i \geq 0$, where Q_i is the i^{th} Milnor primitive in A.

Example 2.2.4 Let $B\mathbb{Z}/2$ denote a classifying space of the group $\mathbb{Z}/2$. The unstable A-module $H^*(B\mathbb{Z}/2)$ is nil-closed indeed, as a graded $\mathbb{F}_2$-algebra $H^*(B\mathbb{Z}/2)$ is freely generated by one generator of degree one.

2.3. Computation of $D_s\Sigma^{-(s+1)}M$, M nil-closed and $s \geq 0$.

2.3.1 To state our result we use the functor $R_s : \mathcal{U} \dashrightarrow \mathcal{U}$, $s \geq 0$, introduction in [5] page 29 (see also [9]) whose main properties are:

(i) The module R_sM is a sub-A-module of $H^*(B(\mathbb{Z}/2)^s) \otimes M$. In particular R_sM is an unstable A-module.

(ii) Let $H^*(B\mathbb{Z}/2) = \mathbb{F}_2[u]$ where u is of degree one. We denote by $L_s = H^*(B(\mathbb{Z}/2)^s)^{GL_s(\mathbb{Z}/2)}$ the Dickson algebra, that is the sub-algebra of $H^*(B(\mathbb{Z}/2)^s)$ of invariants under the natural action of the general linear group $GL_s(\mathbb{Z}/2) = GL((\mathbb{Z}/2)^s)$. The module R_sM is the L_s-module generated by the elements $St_s(x)$, x in M. These elements $St_s(x)$ are defined inductively by :

$$St_0(x) = x \quad , \quad x \in M.$$

$$St_1(x) = \sum_{i=0}^{n} u^{n-i} \otimes Sq^i x \quad , \; x \in M^n.$$

$$St_s(x) = St_1(St_{s-1}(x)) \quad , \; s \geq 1, \; x \in M$$

iii) Let $E_+\mathfrak{S}_{2^s}$ be the disjoint union of a base point and a contractible space on which the symmetric group $\mathfrak{S}_{2^s}$ acts freely. For any pointed space X, we denote by $\mathfrak{S}_{2^s}X$ the quotient of the space $E_+\mathfrak{S}_{2^s} \wedge (X \wedge \ldots. \wedge X)$, X is smashed with itself 2^s times, by the diagonal action of $\mathfrak{S}_{2^s}$ ($\mathfrak{S}_{2^s}$ acts on $X \wedge \ldots\ldots \wedge X$ by permutation of the factors). Let $\Delta_s : B_+(\mathbb{Z}/2)^s \wedge X \dashrightarrow \mathfrak{S}_{2^s} X$ be a "Steenrod diagonal" determined by a bijection between $(\mathbb{Z}/2)^s$and $\{1,2,\ldots.,2^s\}$. The unstable A-module R_sH^*X is the image of $\Delta_s{}^*$ in the modulo 2 cohomology.

2.3.2 Let $\Omega : \mathcal{U} \dashrightarrow \mathcal{U}$ be the left adjoint functor of $\Sigma : \mathcal{U} \dashrightarrow \mathcal{U}$, that is :

$$\mathrm{Hom}_{\mathcal{U}}(M,\Sigma N) = \mathrm{Hom}_{\mathcal{U}}(\Omega M,N)$$

for every unstable A-modules M and N.
We are now ready to state the main result of this paragraph which will be proved in 2.6