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EXOTIC MULTIPLICATIONS ON MORAVA K-THEORIES AND THEIR LIFTINGS

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Abstract. For each prime p and integer n satisfying $0 < n < \infty$, there is a ring spectrum $K(n)$ called the n th Morava K-theory at p. We discuss exotic multiplications upon $K(n)$ and their liftings to certain characteristic zero spectra $\widehat{E(n)}$.

Introduction.

The purpose of this paper is to describe exotic multiplications on Morava's spectrum $K(n)$ and certain "liftings" to spectra whose coefficient rings are of characteristic 0. Many of the results we describe are probably familiar to other topologists and indeed it seems likely that they date back to foundational work of Jack Morava in unpublished preprints. not now easily available. A published source for some of this is the paper of Urs Würgler [**12**]. We only give sketches of the proofs, most of which are straightforward modifications of existing arguments or to be found in [**12**]. For all background information and much notation that we take for granted, the reader is referred to [**1**] and [**7**].

I would like to express my thanks to the organisers of the Luminy Conference for providing such an enjoyable event.

Convention: Throughout this paper we assume that p is an *odd* prime.

§1 Exotic Morava K-theories.

Morava K-theory is usually defined to be a multiplicative complex oriented cohomology theory $K(n)^*(\ )$ which has for its coefficient ring

$$K(n)_* = \mathbf{F}_p[v_n, v_n^{-1}]$$

where $v_n \in K(n)_{2p^n-2}$, and is canonically complex oriented by a morphism of ring spectra

$$\sigma^{K(n)}: BP \longrightarrow K(n)$$

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which on coefficients induces the ring homomorphism

$$\sigma_*^{K(n)} \colon BP_* \longrightarrow K(n)_*$$
$$\sigma_*^{K(n)}(v_k) = \begin{cases} v_n & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Here we have $BP_* = \mathbf{Z}_{(p)}[v_k : k \geq 1]$ with $v_k \in BP_{2p^k-2}$ being the k th *Araki generator*, defined using the formal group sum

$$[p]_{BP}X = \sum_{0\leq k}^{BP} \left(v_k X^{p^k}\right).$$

As a homomorphism of graded rings, we can regard $\sigma_*^{K(n)}$ as a quotient homomorphism

$$\sigma_*^{K(n)} \colon v_n^{-1}BP_* \longrightarrow v_n^{-1}BP_*/\mathcal{M}_n \cong K(n)_*$$

where $\mathcal{M}_n = (v_k : 0 \leq k \neq n) \lhd v_n^{-1}BP_*$ is a maximal graded ideal of the ring $v_n^{-1}BP_*$. Thus we can interpret $K(n)_*$ as a (graded) residue field for this maximal ideal.

Clearly this ideal $\mathcal{M}_n$ is not the only such maximal ideal and we can reasonably look at other examples and ask if the associated quotient (graded) fields occur as coefficient rings for cohomology theories is an analogous fashion. Notice that $\mathcal{M}_n$ contains the invariant prime ideal $I_n = (v_k : 0 \leq k \leq n-1)$ and the formal group law $F^{K(n)}$ therefore has height n. One way to construct $K(n)$-theory is by using Landweber's Exact Functor Theorem (LEFT) [**6**] in its modulo I_n version [**14**]; this allows us to make the definition

$$K(n)^*(\ ) = K(n)_* \otimes_{P(n)_*} P(n)^*(\ )$$

on the category of finite CW spectra $\mathbf{CW^f}$, where $P(n)$ is the spectrum for which

$$P(n)_* = BP_*/I_n.$$

We thus concentrate on maximal ideals $\mathcal{M}' \lhd v_n^{-1}BP_*$ containing the ideal I_n. We then have

THEOREM (1.1). *Let $\mathcal{M}' \lhd v_n^{-1}BP_*$ be a maximal (graded) ideal containing I_n. Then there is a unique multiplicative cohomology theory*

$K(\mathcal{M}')^*(\ )$, defined on $\mathbf{CW^f}$, for which there is a multiplicative natural isomorphism

$$K(\mathcal{M}')^*(\ ) \cong K(\mathcal{M}')_* \otimes_{P(n)_*} P(n)^*(\ )$$

and where $K(\mathcal{M}')_* = v_n^{-1}BP_*/\mathcal{M}'$ is the coefficient ring.

The proof is immediate using LEFT.

Of course, if there is an isomorphism of graded rings, $K(\mathcal{M}')_* \cong K(\mathcal{M}'')_*$, then we need to decide if the two theories arising from $\mathcal{M}'$ and $\mathcal{M}''$ can be naturally equivalent.

THEOREM (1.2). *Let $\mathcal{M}', \mathcal{M}'' \lhd v_n^{-1}BP_*$ be graded maximal ideals containing I_n and let $f\colon K(\mathcal{M}')_* \longrightarrow K(\mathcal{M}'')_*$ be an isomorphism of graded rings. Then there is a natural isomorphism of multiplicative cohomology theories on $\mathbf{CW^f}$,*

$$\widetilde{f}\colon K(\mathcal{M}')^*(\ ) \longrightarrow K(\mathcal{M}'')^*(\ )$$

extending f if and only if the the formal group laws $f_*F^{v_n^{-1}BP_*/\mathcal{M}'}$ and $F^{v_n^{-1}BP_*/\mathcal{M}''}$ are strictly isomorphic over the ring $K(\mathcal{M}'')_*$.

The main observation required to prove this result is that these two formal group laws are associated to two complex orientations induced by the composite of the morphisms of ring spectra $BP \longrightarrow v_n^{-1}BP \longrightarrow K(\mathcal{M}'')$.

COROLLARY (1.3). *The theories $K(\mathcal{M}')^*(\ )$ and $K(\mathcal{M}'')^*(\ )$ are representable by ring spectra $K(\mathcal{M}')$ and $K(\mathcal{M}'')$, which are unique up to canonical equivalence in the stable category. Moreover, $K(\mathcal{M}')$ and $K(\mathcal{M}'')$ are equivalent as ring spectra if and only if the formal group laws $f_*F^{v_n^{-1}BP_*/\mathcal{M}'}$ and $F^{v_n^{-1}BP_*/\mathcal{M}''}$ are strictly isomorphic over the ring $K(\mathcal{M}'')_*$.*

Let us now consider such ring spectra $K(\mathcal{M}')$ where $K(\mathcal{M}')_* \cong K(n)_*$ as graded rings. By a result from [**12**] (see also [**5**]) these are precisely the ring spectra having the homotopy type of $K(n)$ (not necessarily multiplicatively). Thus, such ring spectra are classified to within equivalence as ring spectra by the set of maximal ideals $\mathcal{M}'$ modulo strict isomorphism of the associated formal group laws over $K(\mathcal{M}')_*$. We call the multiplicative cohomology theory associated to such a ring spectrum an *exotic Morava K-theory*.

Let us consider such a spectrum $K(\mathcal{M}')$, where $K(\mathcal{M}')_* \cong K(n)_*$ as graded rings. Then we have the following modification of a result of [**13**], THEOREM (1.4). *As an algebra over $K(n)_*$, we have*

$$K(\mathcal{M}')_*(K(\mathcal{M}')) \cong K(\mathcal{M}')_*(t'_k : k \geq 1) \otimes \Lambda_{K(n)_*}(a'_0, \ldots, a'_{n-1})$$

where $|t'_k| = 2p^k - 2$, $|a'_k| = 2p^k - 1$, and there are polynomial relations of the form

$${t'_k}^{p^n} - v_n^{(p^k-1)/(p-1)} t'_k = h_k(t'_1, \ldots, t'_{k-1})$$

over $K(n)_*$.

The symbol $\Lambda_{K(n)_*}$ denotes an exterior algebra over $K(n)_*$ on the indicated generators.

To prove this result, we rework the proof for the case of $K(n)$ (see [13], [7]) and define the generators t'_k by using the identity

$$\sum_{\substack{r\geq 0\\ s\geq n}}^{K(\mathcal{M}')} (v'_s {t'_r}^{p^s} X^{p^{r+s}}) = \sum_{\substack{r\geq 0\\ s\geq n}}^{K(\mathcal{M}')} (t'_r {v'_s}^{p^r} X^{p^{r+s}})$$

where

$$[p]_{F^{K(\mathcal{M}')}} X = \sum_{s\geq n}^{K(\mathcal{M}')} (v'_s X^{p^s}).$$

The exterior generators are similarly derived.

We can interpret the algebra

$$K(\mathcal{M}')_*(t'_k : k \geq 1)$$

as representing the strict automorphisms of the group law $F^{K(\mathcal{M}')}$, in a way analogous to the case of $K(n)$ (see [7]).

§2 Liftings of exotic Morava *K*-theories.

Recall that there is a ring spectrum $E(n)$ for which

$$E(n)^*(\ ) \cong E(n)_* \otimes_{BP_*} BP^*(\ )$$

on $\mathbf{CW^f}$. Here we have

$$E(n)_* = v_n^{-1} BP_*/(v_{n+k} : k \geq 1).$$

We showed in joint work with Urs Würgler (see [4]) that the *Noetherian completion* $\widehat{E(n)}$ of $E(n)$, characterised by the formula

$$\widehat{E(n)}^*(\ ) = \varprojlim_k \left(E(n)^*(\ )/I_n^k E(n)^*(\ )\right)$$