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CONTINUOUS COHOMOLOGY AND REAL HOMOTOPY TYPE II

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Introduction.

In our earlier paper "Continuous Cohomology and Real Homotopy Type" [**3**], we studied localization of simplicial spaces at the reals and established an equivalence between the category of free nilpotent differential graded commutative algebras of finite type over the reals and nilpotent simplicial spaces of finite type localized at the reals. In this paper, we extend these results by eliminating the nilpotent condition on the algebraic side, thus proving a conjecture of Sullivan [**8**]. (See Theorem 1.2, Part (iv), below.) The main technical work consists in introducing local coefficients into continuous cohomology, continuous de Rham cohomology, the Serre Spectral Sequence, and the constructions involved in real homotopy type.

We also obtain information about secondary characteristic classes of G foliations in the sense of Haefliger [**1**,3,4,6], namely that when G is compact, the continuous cohomology of the appropriate classifying space injects into the ordinary cohomology. This result is stated and proved at the end of Section 2. (See Proposition 2.5).

Our main results are stated in Section 1. The remainder of the paper is devoted to proving these results.

1. Statements of Results.

We begin by recalling some of the notation and definitions from [**3**].

Let $\mathcal{CA}$ denote the category of differential (degree $+1$), graded, commutative (in the graded sense), locally convex topological algebras with unit over R and $\Delta\mathcal{T}$ the category of compactly generated simplicial spaces. Let Ω^p_q denote the space of C^∞ differential p-forms on the standard q-simplex Δ^q in the C^∞ topology. Then $\Omega^p = \{\Omega^p_q\}$ is in $\Delta\mathcal{T}$, $\Omega_q = \{\Omega^p_q\}$ is in $\mathcal{CA}$, and $\Omega = \{\Omega^p_q\}$ is in $\Delta\mathcal{CA}$. Define contravariant functors $\Delta : \mathcal{CA} \longrightarrow \Delta\mathcal{T}$ and $\mathcal{A} : \Delta\mathcal{T} \longrightarrow \mathcal{CA}$ by

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$\Delta(A)_q = (A, \Omega_q) =$ the simplicial space of algebra mappings $A \to \Omega_q$,

$\mathcal{A}(X)^p = (X, \Omega^p) =$ the vector spaces of continuous simplicial mappings $X \to \Omega^p$.

The simplicial structure on Ω gives $\Delta(A)$ a simplicial structure and the algebra structure on Ω gives one on $\mathcal{A}(X)$. We view $\Delta(A)$ as the *simplicial realization* of A and $\mathcal{A}(X)$ as the *algebra of differential forms* on X.

For $X \in \Delta\mathcal{T}$ and any topological abelian group G, let $C^q(X;G)$ be the space of continuous mappings $u : X_q \to G$ with $u \circ s_i = 0, 0 \leq i \leq q-1$, and define $\delta : C^q(X;G) \to C^{q+1}(X;G)$ by

$$\delta u = \sum_{j=0}^{q+1}(-1)^j u \circ \partial_j$$

Here, s_i, ∂_j denotes the face and degeneracy mappings of X. *The continuous cohomology of X with coefficients in G* is defined by

$$H^*(X;G) = H_*(C^*(X;G);\delta).$$

The usual deRham mapping defines an isomorphism

$$\psi : H^*(\mathcal{A}(X);d) \longrightarrow H^*(X;R) = H^*(X).$$

(See Theorem 2.4 of [**3**].)

We next describe homology of $A \in \mathcal{CA}$ with local coefficients. Suppose L is a finite dimensional Lie algebra which acts on a finite dimensional vector space V via a Lie algebra homomorphism $\gamma : L \to g\ell(V) = \mathrm{Hom}(V,V)$. Let $C^*(L)$ denote the usual cochain algebra on L, that is, $C^p(L)$ is the space of alternating, multilinear functions

$$u : L^p = L \times L \times \cdots \times L \to R$$

with $d : C^p(L) \to C^{p+1}(L)$ given by

$$du(\ell_1, \ldots, \ell_{p+1}) = \sum_{i<j}(-1)^{i+j}u([\ell_i, \ell_j], \ell_1, \ldots, \hat{\ell}_i, \ldots, \hat{\ell}_j, \ldots, \ell_{p+1})$$

For $A \in \mathcal{CA}$, we define L-local coefficients on A as follows. Let $\ell_1, \ldots, \ell_n$ be a basis for L, $\ell_1^*, \ldots, \ell_n^*$ the dual basis for L^*, and suppose $\lambda : C^*(L) \to A$ is a $\mathcal{CA}$ mapping. Define $d_\lambda : A \otimes V \to A \otimes V$ by

$$d_\lambda(a \otimes v) = da \otimes v + (-1)^p \sum_{i=1}^{n} a\lambda(\ell_i^*) \otimes \ell_i v.$$

where $\ell_i v = \gamma(\ell_i)(v)$. It is easy to check that d_λ is independent of the choice of basis, that $d_\lambda^2 = 0$, and that d_λ is functorial in both A and V. Let $H_*(A;V_\lambda) = H_*(A \otimes V, d_\lambda)$.

Remark 1.1. If $A = C^*(L), \lambda =$ identity, $\gamma : L \to g\ell(V)$, and $J : C^*(L) \otimes V \to C^*(L; V)$ is the standard isomorphism, then $Jd_\lambda = d_\gamma J$ where $d_\gamma : C^p(L; V) \to C^{p+1}(L; V)$ is given by

$$d_\gamma \omega = d\omega + \gamma \wedge \omega.$$

Here, γ is considered as a $g\ell(V)$-valued 1-form on L and the wedge product $\gamma \wedge \omega$ is defined using the action of $g\ell(V)$ on V.

Suppose now that $A \in \mathcal{CA}$ is free and of finite type; that is, A is the tensor product of a polynomial algebra on even dimensional generators with an exterior algebra on odd dimensional generators and each A^j is a finite dimensional vector space, $j \geq 0$. According to Proposition 7.11 of [**2**], we can find a basis $t_1, \ldots, t_n$ for A^1 such that, for $1 \leq i \leq m$,

$$dt_i = \sum_{\substack{1 \leq i < j \leq m \\ j < k}} a_i^{jk} t_j t_k$$

and for $m < i \leq n, dt_i$ is a polynormal generator for A. One easily sees that, if A and B are free and of finite type, then $\Delta(A \otimes B) = \Delta(A) \times \Delta(B)$ and if $A = R[x, y]$ with $dx = y$, then ΔA is contractible in $\Delta\mathcal{T}$. Hence, up to homotopy type, $\Delta(A)$ is unchanged by dividing A by the ideal generated by $\{t_i, dt_i \mid i > m\}$. Henceforth, we include the condition $n = m$ in the notion of free and of finite type.

Given A as above, let L be the dual vector space to A^1 and let $\alpha_1, \ldots, \alpha_m$ be the basis for L dual to $t_1, \ldots, t_m$. Then L is a Lie algebra with

$$[\alpha_j, \alpha_k] = 2 \sum_{i=1}^{m} a_i^{jk} \alpha_i.$$

The inclusion

$$\lambda : C^*(L) \simeq R[t_1, \ldots, t_m] \subset A$$

defines L-local coefficients on A. As in [**3**], we define $i : A \to \mathcal{A}(\Delta(A))$ by $i(a)(u) = u(a)$. Then $i\lambda : C^*(L) \to \mathcal{A}(\Delta(A))$ defines L-local coefficients on $\mathcal{A}(\Delta(A))$. Finally, if $A^{(1)}$ denotes the subalgebra of A generated by $t_1, \ldots, t_m$, then $C^*(L)$ is naturally isomorphic to $A^{(1)}$.

The following result is stated in [**8**] as "Theorem" 8.1.

THEOREM 1.2. *Suppose $A \in \mathcal{CA}$ is free of finite type, and that $A^{(1)} = C^*(L)$ as above.*

(i) Let $G = G_A$ be the connected, simply connected Lie group with $L(G) = L$. Then

$$\begin{aligned} \pi_i(\Delta A^{(1)}) &\simeq G \quad \text{for } i = 1, \\ &\simeq \pi_i(G) \quad \text{for } i > 1. \end{aligned}$$

(ii) Let V be a finite dimensional vector space on which L acts and $\lambda : C^*(L) \to A$ the inclusion map. Then the mapping $i : A^{(1)} \to \mathcal{A}(\Delta(A^{(1)}))$ induces an isomorphism

$$i_* : H_*(A^{(1)}; V_\lambda) \to H_*(\mathcal{A}(\Delta(A^{(1)})); V_{i\lambda}) \simeq H^*(\Delta A^{(1)}; V_{i\lambda}).$$

(iii) Let $\tilde{A}$ be the quotient algebra of A by the ideal generated by $A^{(1)}$. Then $A^{(1)} \subset A$ induces a fibre map $\Delta(A) \to \Delta(A^{(1)})$ with fibre $\Delta(\tilde{A}) \subset \Delta(A)$. (Note that $\tilde{A}$ is free, nilpotent, and of finite type and hence the homotopy type of $\Delta(\tilde{A})$ is described in [3].)

(iv) For any action of L on V, the mapping $i : A \to \mathcal{A}(\Delta(A))$ induces isomorphisms

$$i_* : H_*(A; V_\lambda) \longrightarrow H_*(\mathcal{A}(\Delta(A)); V_{i\lambda}) \simeq H^*(\Delta(A); V_{i\lambda}).$$

In [8], Sullivan gives a very brief sketch of (i) and (iii) and, asserts that (ii) is "a reformulation of the theorem of Van Est". No proof is given for (iv). We give a detailed proof of (iv) in general and of (ii) when $G = G_A$ in the universal cover of a compact group. Actually (i) follows from Proposition 2.4 and (iii) follows from results of Section 5. In Section 2, we give an analysis of $\Delta(C^*(L))$ (see Theorem 2.3 and Proposition 2.4). Section 3 deals with local coefficients and the de Rham theorem. Proposition 2.4, the de Rham theorem with local coefficients, and an unpublished result of Graeme Segal are used in Section 4 to prove (ii) when G is the universal cover of a compact group. The result of Segal is that the continuous cohomology and the ordinary cohomology of the singular complex of a CW complex are isomorphic. We give Segals proof in Section 7.

The development of the proof of (iv) is as follows: Suppose $A \in \mathcal{CA}$ is free and of finite type. Then $A = UA^{(n)}$ where $A^{(0)} = R$ and $A^{(n)} = A^{(n-1)}[x_1^{(n)}, \ldots, x_k^{(n)}]$ and the $x_i^{(n)}$ have dimension n. We compute $H^*(\Delta(A); V_{i\lambda})$ by computing $H^*(\Delta(A^{(n)}); V_{i\lambda})$ using induction on n. In Section 6, we use Proposition 2.4 to prove Theorem 5.3, namely that

$$\Delta(A[x_1, \ldots, x_k]) \to \Delta(A)$$

is a fibration with fibre $\Delta(R[x_1, \ldots, x_k])$. In Section 6 we develop the Serre Spectral Sequence for continuous cohomology with local coefficients and apply it to the above fibration to prove the inductive step in the proof of (iv).

Recall that, for $A, B \in \mathcal{CA}$, a function complex $\mathcal{F}(A, B) \in \Delta\mathcal{T}$ was defined in [3] (following [2]) by $\mathcal{F}(A, B)_q = (A, \Omega_q \otimes B)$, the space of continuous differential graded algebra mappings from A to $\Omega_q \otimes B$. If A and $B \in \mathcal{CA}$ are free and of finite type and $h : A^{(1)} \to B^{(1)}$ is a map in $\mathcal{CA}$, define $\mathcal{F}(A, B; h)$ to be the simplicial subspace of $\mathcal{F}(A, B)$ whose q simplicies are maps $u : A \to \Omega_q \otimes B$ which give a commutative