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M. C. CRABB

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THE FULLER INDEX AND $\mathbb{T}$-EQUIVARIANT STABLE HOMOTOPY THEORY

by M.C. CRABB

0. Introduction

In a remarkable paper [8], published more than twenty years ago, Fuller introduced an index which counts periodic orbits of smooth flows. Let w be a smooth vector field defined on a (finite-dimensional) closed manifold X and $\theta_t : X \to X$, $(t \in \mathbb{R})$, the corresponding flow (so that $\theta_0 = 1$ and $\dot{\theta}_t = w(\theta_t)$, where the dot denotes differentiation). Suppose that U_1 is an open subspace of $(0,\infty) \times X$ such that the set

$$(0.1) \qquad F = \{(T,x) \in U_1 \mid \theta_T x = x\}$$

is compact. To such a field w and open set U_1, Fuller associates a $\mathbb{Q}$-valued index, which vanishes if F is empty.

In 1985, Ize [10] and Dancer [6] observed, independently, that the natural setting for Fuller's index is $\mathbb{T}$-equivariant homotopy theory, $\mathbb{T}$ being the circle group $\mathbb{R}/\mathbb{Z}$. My purpose here is to describe their work from the viewpoint of algebraic topology using the standard methods of equivariant fixed-point theory over a base.

The relevance of the $\mathbb{T}$-equivariant theory is not hard to see. Indeed, if $(T,x) \in F$, (0.1), then the compactness of F implies that $(T,\theta_t x) \in F$ for all $t \in \mathbb{R}$ and, also, that x is not a stationary point of the flow $(w(x) \neq 0)$. So we can define a fixed-point-free circle action on F by:

$$(0.2) \qquad [t].(T,x) = (T,\theta_{tT}x),$$

for $t \in \mathbb{R}$, $[t] = t + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. The Fuller index is, in a sense to be made precise, a count of this set F, with the fixed-point-free $\mathbb{T}$-action, over the base $(0,\infty)$.

Each point $(T,x) \in F$ determines a periodic solution $\gamma(t) = \theta_t x$, of period T, of the differential equation:

$$\dot{\gamma} - w(\gamma) = 0, \tag{0.3}$$

or, by re-scaling, a solution $\alpha\colon \mathbb{R} \to X$, $\alpha(t) = \theta_{tT}x$, of period 1 of:

$$\dot{\alpha} - Tw(\alpha) = 0. \tag{0.4}$$

It is convenient to make no distinction in notation between a map $\alpha\colon \mathbb{R} \to X$ of period 1 and the corresponding loop $\alpha\colon \mathbb{R}/\mathbb{Z} = \mathbb{T} \to X$. Then we can think of solutions of (0.4) as zeros of a vector field on the infinite-dimensional manifold $M = LX$ of smooth loops $\mathbb{T} \to X$ in the following way. (See, for example, Atiyah [1] and Bismut [3].)

Recall that the tangent space $\tau_\alpha M$ at a point $\alpha \in M$, $\alpha\colon \mathbb{T} \to X$, can be identified with the space of smooth sections of $\alpha^*\tau X$ over $\mathbb{T}$. So we can regard $t \mapsto w(\alpha(t))$ as a tangent vector $w(\alpha) \in \tau_\alpha M$, and the vector field w on X thus defines a vector field, of the same name, on M. The circle acts on M by rotating loops: $([t].\alpha)(u) = \alpha(t+u)$, for $t,u \in \mathbb{R}$. This $\mathbb{T}$-action has a generating vector field, s say, given by differentiation:

$$s(\alpha) = \dot{\alpha}. \tag{0.5}$$

The zero-set of s, or the fixed subspace $M^{\mathbb{T}}$, is the space X of constant loops.

Now we have a family $v_T = s - Tw$, $T > 0$, of $\mathbb{T}$-equivariant vector fields on M, parametrized by $(0,\infty)$, and the zero-set of v_T is precisely the set of solutions of (0.4). Let U_∞ be the open subset $\{(T,\alpha) \in (0,\infty) \times M \mid (T,\alpha(t)) \in U_1 \text{ for all } t \in \mathbb{R}\}$ of $(0,\infty) \times M$. Then the zero-set

$$\{(T,\alpha) \in U_\infty \mid v_T(\alpha) = 0\} \tag{0.6}$$

is equivariantly homeomorphic to F, (0.1) and (0.2), and so compact.

The problem is to define an index for such a family of vector fields v_T with compact zero-set in some open subspace of $(0,\infty) \times M$. There are technical difficulties in infinite-dimensions: in order to apply the Leray-Schauder theory (as described in [9], for example) it is necessary to replace v_T by a "renormalized" field satisfying a certain compactness condition. This analysis, which is joint work with A.J.B. Potter, will appear elsewhere. In this paper, following Dancer [6], I shall concentrate on the analogous finite-dimensional problem, which illustrates all the algebraic topological features of the Fuller index. This is done in Section

2. Section 1 reviews the, now standard, equivariant index theory over a base for zeros of vector fields and fixed-points of maps, developed by Dold, Becker and Gottlieb in the mid seventies.

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1. The vector-field index

This section contains an outline, in a form tailored to the applications, of the Poincaré-Hopf index theory for vector fields. Whilst this theory can be viewed as a special case of the Lefschetz fixed-point theory, it seems worth maintaining a conceptual distinction. We confine the discussion to the non-equivariant theory. The modifications needed to produce the G-equivariant index theory, for a compact Lie group G (acting smoothly on manifolds), are technical rather than conceptual. The treatment here is strongly influenced by the work of Dold (as in [7] and the references there). A detailed account can be found in [12].

Consider first a (continuous) vector field v defined on an open subset U of a (finite-dimensional) Euclidean space V, and suppose that the zero-set

$$\mathrm{Zero}(v) = \{x \in U \mid v(x) = 0\} \tag{1.1}$$

is compact. The basic index, $\tilde{I}(v,U)$ say, is a stable map $S^0 \to U_+$ (where the subscript "+" denotes adjunction of a disjoint basepoint). It is defined by an explicit geometric construction in the style of Pontrjagin-Thom as follows.

We can regard the vector field v simply as a map $v: U \to V$. Let $N \subseteq V$ be an open neighbourhood of Zero(v) such that $\overline{N}$ is compact and $\overline{N} \subseteq U$, and choose a (finite) open ball B, centre 0, in V so small that $v(x) \notin B$ for all $x \in \overline{N} - N$. Using a superscript "+" for one-point-compactification, we define a map $q: V^+ \to (V/(V-B)) \wedge U_+$, by $q(x) = [v(x),x]$ if $x \in \overline{N}$, $q(x) = *$ (basepoint) if $x \notin N$. Then, identifying $V/(V-B) = B^+$ with V^+ by radial extension, we obtain a well-defined homotopy class $V^+ \to V^+ \wedge U_+$, which represents the stable map $\tilde{I}(v,U): S^0 \to U_+$.

1.2 REMARK. At this level the vector-field and fixed-point problems are indistinguishable. The construction just described defines the Lefschetz fixed-point index of the map $f\colon U \to V$ given by $f(x) = x - v(x)$. The zeros of v are the fixed-points of f.

Two fundamental properties of the index are evident from the construction.

1.3 PROPERTIES OF THE INDEX.

(a) <u>Suppose that</u> U' <u>is an open subset of</u> U <u>containing</u> $\mathrm{Zero}(v)$. <u>Then</u> $\tilde{I}(v,U) = i_+ \circ \tilde{I}(v,U')$, <u>where</u> i_+ <u>is the inclusion</u>.

(b) <u>Suppose that</u> U <u>is a disjoint union of open subsets</u> U_1 <u>and</u> U_2. <u>Then</u> $\tilde{I}(v,U) = i_+^1 \circ \tilde{I}(v,U_1) + i_+^2 \circ \tilde{I}(v,U_2)$, <u>where</u> i_+^1 <u>and</u> i_+^2 <u>are the respective inclusions of</u> U_1 <u>and</u> U_2 <u>in</u> U.

Composing $\tilde{I}(v,U)$ with the map $S^0 \to U_+$ which collapses U to a point, we obtain a stable map $S^0 \to S^0$ or, in other words, an element, $I(v,U)$ say, of the stable cohomotopy ring $\omega^0(*)$. (The symbol "ω" is used for unreduced stable homotopy.) This class $I(v,U)$ is the traditional Poincaré-Hopf index. Of course, in this case it is just an integer and determined by $\mathbb{Z}$-cohomology. The definitions have been formulated in this way so as to generalize directly to the equivariant bundle theory.

Next we recall the computation of the index for a field with isolated zeros. Suppose that $\mathrm{Zero}(v)$ lies in the interior of the unit disc $D(V)$ in V and that $D(V) \subseteq U$. Then $I(v,U) \in \omega^0(*)$ is the stable homotopy class represented by the map of spheres:

$$S(V) \to S(V) : x \mapsto \frac{1}{|v(x)|} v(x), \tag{1.4}$$

(so in this case the classical degree). With the additivity of the index, (1.3)(b), this determines $I(v,U)$ when $\mathrm{Zero}(v)$ is discrete.

In the differentiable case, the index of a non-degenerate zero lies in the image of the J-homomorphism. Suppose that the vector field v is continuously differentiable (C^1) with $\mathrm{Zero}(v) = \{0\}$ and the derivative $(Dv)(0)\colon V \to V$ invertible. Then $(Dv)(0)$ defines an element "sign det" of $KO^{-1}(*) = \mathbb{Z}/2$, and $I(v,U)$ is the image of this class under the J-homomorphism

$$J : KO^{-1}(*) \to \omega^0(*)^{\cdot} \subseteq \omega^0(*) \tag{1.5}$$

to the group of units $\omega^0(*)^{\cdot} = \{\pm 1\}$ in the stable cohomotopy ring.