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A Geometric Interpretation
of Lannes' Functor T.

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1. Introduction. In this note we are concerned with a question raised by [Lannes 2.3]. In what follows R will denote a finite field of the form $\mathbf{Z}/p\mathbf{Z}$, homology and cohomology are always taken with coefficient in R and denoted by H_*X etc. For a space X let $\{R_sX\}_s$ denote the Bousfield-Kan localization tower. We denote by $B\tau$ the classifying space of the underlying abelian group of R. Let P_sX denote the s-Postnikov section of X. By a "space" we mean a Kan complex or a C.W. complex.

1.1 Theorem: *If $H^iX < \infty$ for all $i \geq 0$, then $TH^*X \cong \varinjlim_s H^*(P_sR_sX)^{B\tau}$, where T is Lannes' functor* (see below). *If, in addition, X is nilpotent then $TH^*X \cong \varinjlim$*

$H^*(P_sX)^{B\tau} \cong \varinjlim H^*(P_sR_\infty X)^{B\tau}$

The proof of this theorem yields a new proof for Lannes theorem 1.5 below that essentially asserts 1.1 for dimension zero and was a the motivation for his question [Lannes 2.3]. The proof of theorem is based on the following technical proposition:

1.2 Proposition: *Let $G \to E \to B$ be a principal fibration where G is a (topological or simplicial) group. Assume that in each dimension the R-cohomology of the mapping spaces E^{B_t} and B^{B_t} is finite. Then if the relation $TH^*W \cong H^*W^{B_t}$ is satisfied by $W = E$ and $W = B$ then it is also satisfied by $W = G$.*

Remark: The finiteness assumption, noted by the referee, is necessary in order to use cohomological Eilenberg-Moore spectral sequence.

Corollary: *If W is a nilpotent space of finite type with $\pi_iW = 0$ for $i >> 0$ or a R-localization thereof then*

$$TH^*W \cong H^*W^{B\tau}.$$

Further, as a direct corollary of 1.1 and 9.3 of [Bousfield] one gets the following interesting special case due to Lannes [4].

1.3 Corollary. *Let $H^iX < \infty$ for all $i \geq 0$ for nilpotent space X of finite type. Assume that a given algebraic component T_cH^*X of TH^*X is finite in all dimensions and vanishes in dimension one. Then* $T_cH^*X \cong H^*X_c^{B\tau} \cong H^*(R_\infty X)_c^{B\tau}$

where $X_c^{B\tau}$ is the corresponding component.

Another example where the main result (1.1) implies a result on $H^*map(B\tau, X)$ is when the latter has a finite homotopy group in each dimension.

1.4 Corollary: *Let X be nilpotent space of finite type with $\pi_1 X$ finite. Assume that for a given $f : B\tau \to X$ the groups $\pi_i map(B\tau, P_nX)_{f_*}$ are finite for all $i, n \geq 0$. Then $H^*map(B\tau, X) \cong T_cH^*X$ where T_c is algebraic component of T that corresponds to f.*

The referee also notes that theorem 1.1 gives a new proof of the following result [Lannes, 0.4].

1.5 Corollary: *If Y is a nilpotent space with $H^n(Y, R)$ finite for all n, then the natural map*

$$[Bt, Y] \cong [Bt, R_\infty Y] \to Hom_K(H^*Y, H^*B_t)$$

is an isomorphism of profinite sets.

Proof: This follows directly from 1.1 above in light of the algebraic fact [Lannes 3.5] and the old result of [Dror] about the tower $R_s Y$.

The authors would like to thank W. Dwyer for his suggestion to consider the tower $R_s X$ as a starting point for a geometric interpretation of T, and to H. Miller for several useful discussions. The authors would also like to thank the referee for his careful reading and for correcting a non-fatal error in an earlier version of this paper. The referee notes that if one considers the fibration $\Omega X \to * \to X$ for X being the infinite wedge of $\mathbb{R}P^\infty/\mathbb{R}P^{2n+2}$ over the integers, the formula in 1.2 holds for $W = \Omega X$ but not for X itself. Therefore one cannot turn around 1.2 to conclude that either E or B satisfy the property $TH^*W = H^*W^{B_t}$, assuming the other two spaces do.

2 First examples.

Let $\mathcal{U}$ denote the category of unstable modules over the algebra $\mathcal{A}$ of Steenrod reduced powers relative to a prime $p = char\ R$. Let $\mathcal{K}$ denote the category of unstable $\mathcal{A}$-algebra. In [Lannes] a left adjoint T is defined to the functor $- \otimes H^*B\tau$, where the latter is taken either as a functor from $\mathcal{U}$ to itself or from $\mathcal{K}$ to itself. If one regards an element $A \in \mathcal{K}$ as an element of $\mathcal{U}$, the value of T does not change. Thus the defining property of T is $hom_C(TM, N) = hom_C(M, N \otimes H^*B\tau)$ where C is either $\mathcal{U}$ or $\mathcal{K}$.

2.1 Three basic properties [Lannes]: (i) T is exact. (ii) T commutes with tensor products. (iii) T commutes with direct limits.

2.2 Motivation: It can be seen from 1.1, 1.2, 1.3 that the construction of T is motivated by attempts to describe the cohomology of $X^{B\tau} \equiv map(B\tau, X)$ in terms of H^*X, when the latter is given as an object in $\mathcal{K}$. Lannes proves the relation between the homotopy class $[B\tau, X]$ and $(TH^*X)^0$ and X as in 1.3, see [Lannes 7.1.1]. [Miller] proves it for $dim X < \infty$.

2.3 Example. It is easy to calculate directly from the adjointness relation that if V is a finite dimensional vector space over R then

$$TH^*K(V,n) \cong \bigotimes_{n \geq i \geq 0} H^*K(V,i) \cong H^*map(B\tau, K(V,n)).$$

Here $map(X,Y)$ denotes the space of maps $X \to Y$ otherwise denoted by Y^X. Similar calculation holds for a finite products of $K(V_i, n_i)$ with $dim_R V_i < \infty$. However it turns out that for homotopically large space one cannot, in general, interpret TH^*X as the cohomology of $map(B\tau, X)$, (see 2.5 below).

2.4. Example. An important class of spaces on which T behaves nicely are finite Postnikov pieces of nilpotent spaces. The prime examples of such spaces are $K(\mathbf{Z}, n)$ spaces for $n \geq 0$.

Proposition: *For any $n \geq 0$ there is an isomorphism $TH^*K(\mathbf{Z},n) \cong H^*map(B\tau, K(\mathbf{Z},n))$.*

Proof: For $p = 2$ we show by a direct computation that $TH^*K(\mathbf{Z},n) \cong H^* \prod_{i=1}^{[n/2]} K(\mathbf{Z}/2\mathbf{Z},\ 2i)$. For $p > 2$ the argument is similar. Now since $H^*K(\mathbf{Z},n) \cong P(S_q^I \mid I$ admissible with $e_1(I) \geq 2$ and $e\ (I) \leq n-1)$ a map of the algebra $H^*K(\mathbf{Z},n)$ over A is given by the image of the generator in dimension n. Thus

$$hom_{\mathcal{K}}(K(\mathbf{Z},n), K) = ker\beta : K_n \to K_{n+1}$$

where K is any object in $\mathcal{K}$ and β is The Bockstein operation. Now compute:

$$\begin{aligned}
hom_K(TH^*K(\mathbf{Z},n),K) &\cong hom_K(H^*K(\mathbf{Z},n),K\otimes H^*B\tau)\\
&\cong ker\beta : K\otimes H^*B\tau)_n \to (K\otimes H^*B\tau)_{n+1}\\
&\cong ker\beta : \bigoplus_{i+j=n} K_i\otimes H^jB\tau \to \bigoplus_{i+j=n+1} K_i\otimes H^jB\tau\\
&\cong \bigoplus_{\substack{i+j=n\\ j\ \text{even}}} K_i =\\
&\cong hom_K(H^*\prod_{i=1}^{[n/2]} K(\mathbf{Z}(2\mathbf{Z},i),K).
\end{aligned}$$

This together with the adjointness property of T completes the proof. Similarly let $\mathbf{Z}_p$ denotes the p-adic numbers $\mathbf{Z}_p$ = invlim $\mathbf{Z}/p^k\mathbf{Z}$. Then [B - K VI 6.4] one has an R homology equivalence $K(\mathbf{Z},n)\to K(\mathbf{Z}_p,n)$ for all $n\geq 0$. There is a pro-isomorphism on R-homology of $K(\mathbf{Z},n)\to(K(\mathbf{Z}/p^k\mathbf{Z},n))_n$. Therefore

$$H^*K(\mathbf{Z},n)\cong H^*K(\mathbf{Z}_p,n)=\varinjlim_k H^*K(\mathbf{Z}/p^k\mathbf{Z},n)$$

Moreover it follows by a spectral sequence argument that the tower $\{map(B\tau,K(\mathbf{Z}/p^k\mathbf{Z},k))\}_*$ is an R completion tower for the function complex $map(B\tau,K(\mathbf{Z},n))$, since all function complexes involved here are R-nilpotent. Again using comparison of spectral sequences converging to homology we see that there is an R-homology (hence R-cohomology) equivalence $map(B\tau,K(\mathbf{Z},n))\to map(B\tau,K(\mathbf{Z}_p,n))$. Therefore the R-cohomology of the last range is isomorphic to the limit of the R-cohomologies $\varinjlim_k H^*map(B\tau,K(\mathbf{Z}/p^k\mathbf{Z},n))$. But since the functor T commutes with direct limits we get the desired example:

$$TH^*K(\mathbf{Z}_p,n)\cong H^*map(B\tau,K(\mathbf{Z}_p,n)).$$

The second example of $K(\mathbf{Z}_p,n)$ is in reality equivalent to the first using the isomorphism of cohomologies $H^*(B\tau,\mathbf{Z})\cong H^*(B\tau,\mathbf{Z}_p)$. Since the function complexes $hom(B\tau,K(\mathbf{Z},n))$ and $hom(B\tau,K(\mathbf{Z}_p,n))$ are built out of these cohomology groups, the map $\mathbf{Z}\to\mathbf{Z}_p$ induces a homotopy equivalence between them. Now since $TH^*K(\mathbf{Z},n)\cong TH^*K(\mathbf{Z}_p,n)$ one satisfies 2.4 if and only if the other does.