Astérisque

WILLIAM G. DWYER
CLARENCE W. WILKERSON

Spaces of null homotopic maps

Astérisque, tome 191 (1990), p. 97-108
<http://www.numdam.org/item?id=AST_1990__191__97_0>

Spaces of Null Homotopic Maps

WILLIAM G. DWYER AND CLARENCE W. WILKERSON

§1. INTRODUCTION

In 1983 Haynes Miller [7] proved a conjecture of Sullivan and used it to show that if π is a locally finite group and X is a simply connected finite dimensional CW-complex then the space of pointed maps from the classifying space $B\pi$ to X is weakly contractible, ie. $\mathrm{Map}_*(B\pi, X) \simeq *$. This result had immediate applications. Alex Zabrodsky [**11**] used it to study maps between classifying spaces of compact Lie groups. McGibbon and Neisendorfer [**6**] applied Miller's theorem to answer a question of Serre; they proved that if X is a simply connected finite dimensional CW-complex with $\tilde{H}^*(X, \mathbf{F}_p) \neq 0$ then there are infinitely many dimensions in which $\pi_*(X)$ has p-torsion.

The goal of this note is to use the functor T^V of [**2**] to generalize Miller's theorem and some of its corollaries to a large class of infinite dimensional spaces (see [**5**] for closely related earlier work in this direction). This generalization comes at the expense of working with one component of the function complex $\mathrm{Map}_*(B\pi, X)$ at a time.

Fix a prime number p.

THEOREM 1.1. *Let π be a locally finite group and X a simply connected p-complete space. Assume that $H^*(X, \mathbf{F}_p)$ is finitely generated as an algebra. Then the component of* $\mathrm{Map}_*(B\pi, X)$ *which contains the constant map is weakly contractible.*

REMARK: There is a standard way [**7**, 1.5] to relax the assumption in 1.1 that X is p-complete.

Theorem 1.1 is actually a special case of a more general assertion. Recall that an unstable module M over the mod p Steenrod Algebra $\mathbf{A}_p$ is said to be *locally finite* [4] if each element $x \in M$ is contained in a finite $\mathbf{A}_p$ submodule. If R is a connected unstable algebra over $\mathbf{A}_p$ then the *augmentation ideal* $I(R)$ is by definition the ideal of positive-dimensional elements and the *module of indecomposables* $Q(R)$ is the unstable $\mathbf{A}_p$ module $I(R)/I(R)^2$. An unstable algebra R over $\mathbf{A}_p$ is of *finite type* if each R^k is finite-dimensional as an $\mathbf{F}_p$ vector space.

Both authors were supported in part by the National Science Foundation. The first author would like to thank the University of Chicago Mathematics Department for its hospitality during the course of this work.

THEOREM 1.2. *Let π be a locally finite group and X a simply connected p-complete space such that $H^*(X, \mathbf{F}_p)$ is of finite type. Assume that the module of indecomposables $Q(H^*(X, \mathbf{F}_p))$ is locally finite as a module over $\mathbf{A}_p$. Then the component of $\mathrm{Map}_*(B\pi, X)$ which contains the constant map is weakly contractible.*

REMARK: Theorem 1.1 does in fact follow from Theorem 1.2, since if $H^*(X, \mathbf{F}_p)$ is finitely generated as an algebra then $Q(H^*(X, \mathbf{F}_p))$ is a *finite* $\mathbf{A}_p$ module.

REMARK: Theorem 1.2 has a converse, at least if $p = 2$ (see Theorem 3.2). There is also a generalization of 1.2 that deals with other components of the mapping space $\mathrm{Map}_*(B\pi, X)$ (see Theorem 4.1) but for this generalization it is necessary to assume that π is an elementary abelian p-group.

Given 1.2, the arguments of [**6**] go over more or less directly and lead to the following result. A CW-complex is of *finite type* if it has a finite number of cells in each dimension.

THEOREM 1.3. *Suppose that X is a two-connected CW-complex of finite type. Assume that $\tilde{H}^*(X, \mathbf{F}_p) \neq 0$ and that $Q(H^*(X, \mathbf{F}_p))$ is locally finite as a module over $\mathbf{A}_p$. Then there exist infinitely many k such that $\pi_k(X)$ has p-torsion.*

REMARK: The example of CP^∞ shows that it would not be enough in Theorem 1.3 to assume that X is 1-connected.

Some instances of 1.3 were previously known; for instance, if $X = BG$ for G a suitable compact Lie group then the conclusion of 1.3 can be obtained by applying [**6**] to the loop space on X. However, Theorem 1.3 applies in many previously inaccessible cases; for example, it applies if X is the Borel construction $EG \times_G Y$ of the action of a compact Lie group G on a finite complex Y or if X is a quotient space obtained from such a Borel construction by collapsing out a skeleton.

We first noticed Theorem 1.1 as part of our work [**1**] on calculating fragments of T^V with Smith theory techniques. The proof of 1.1 given here does not use the localization approach of [**1**]; it is partly for this reason that the proof generalizes to give 1.2.

Organization of the paper. Section 2 recalls some properties of the functor T^V. In §3 there is a proof of 1.2 and in §4 a generalization of 1.2 to other components of the mapping space. Section 5 uses the ideas of [**6**] to deduce 1.3 from 1.2.

Notation and terminology. The prime p is fixed for the rest of the paper; all unspecified cohomology is taken with $\mathbf{F}_p$ coefficients. The symbol $\mathcal{U}$ (resp. $\mathcal{K}$) will denote the category of unstable modules (resp. algebras) [**2**] over $\mathbf{A}_p$. If $R \in \mathcal{K}$ then $\mathcal{U}(R)$ (resp. $\mathcal{K}(R)$) will stand for the category of objects of $\mathcal{U}$ (resp. $\mathcal{K}$) which are also R-modules (resp. R-algebras) in a compatible way [**1**].

For a pointed map $f : K \to X$ of spaces we will let $\mathrm{Map}_*(K, X)_f$ denote the component of the pointed mapping space $\mathrm{Map}_*(K, X)$ containing f. The component of the *unpointed* mapping space containing f is $\mathrm{Map}(K, X)_f$.

§2 The functor T^V

Let V be an elementary abelian p-group, ie., a finite-dimensional vector space over $\mathbf{F}_p$, and H^V the classifying space cohomology H^*BV. Lannes [**2**] has constructed a functor $T^V : \mathcal{U} \to \mathcal{U}$ which is left adjoint to the functor given by tensor product (over $\mathbf{F}_p$) with H^V and has shown that T^V lifts to a functor $\mathcal{K} \to \mathcal{K}$ which is similarly left adjoint to tensoring with H^V.

PROPOSITION 2.1 [**2**]. *For any object R of $\mathcal{K}$ the functor T^V induces functors $\mathcal{U}(R) \to \mathcal{U}(T^V(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T^V(R))$. The functor T^V is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a natural isomorphism*

$$T^V(M \otimes_R N) \cong T^V(M) \otimes_{T^V(R)} T^V(N)$$

Now suppose that $\gamma : R \to H^V$ is a $\mathcal{K}$-map. The adjoint of γ is a map $T^V(R) \to \mathbf{F}_p$ or in other words a ring homomorphism $\hat{\gamma} : T^V(R)^0 \to \mathbf{F}_p$. For $M \in \mathcal{U}(R)$, let $T^V_\gamma(M)$ be the tensor product $T^V(M) \otimes_{T^V(R)^0} \mathbf{F}_p$, where the action of $T^V(R)^0$ on $\mathbf{F}_p$ is given by $\hat{\gamma}$. Note that $T^V_\gamma(R) \in \mathcal{K}$.

PROPOSITION 2.2 [**1**, 2.1]. *For any $\mathcal{K}$-map $\gamma : R \to H^V$ the functor $T^V_\gamma(-)$ induces functors $\mathcal{U}(R) \to \mathcal{U}(T^V_\gamma(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T^V_\gamma(R))$. The functor T^V_γ is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a natural isomorphism*

$$T^V_\gamma(M \otimes_R N) \cong T^V_\gamma(M) \otimes_{T^V_\gamma(R)} T^V_\gamma(N).$$

The following proposition is a straightforward consequence of the above two.

LEMMA 2.3. *Suppose that $\alpha : R_1 \to R_2$ and $\beta : R_2 \to H^V$ are morphisms of $\mathcal{K}$, and let $\gamma : R_1 \to H^V$ denote the composite $\beta \cdot \alpha$.*

(1) *If α is a surjection and $M \in \mathcal{U}(R_2)$ is treated via α as an object of $\mathcal{U}(R_1)$, then the natural map $T^V_\gamma(M) \to T^V_\beta(M)$ is an isomorphism.*
(2) *If $M \in \mathcal{U}(R_1)$ then the natural map $T^V_\beta(R_2) \otimes_{T^V_\gamma(R_1)} T^V_\gamma(M) \to T^V_\beta(R_2 \otimes_{R_1} M)$ is an isomorphism.*

There is a natural map $\lambda_X : T^V(H^*X) \to H^* \operatorname{Map}(BV, X)$ for any space X. If $g : BV \to X$ is a map which induces the cohomology homomorphism $\gamma : H^*X \to H^V$ then λ_X passes to a quotient map

$$\lambda_{X,g} : T^V_\gamma(H^*X) \to H^* \operatorname{Map}(BV, X)_g \, .$$

A lot of the geometric usefulness of T^V is explained by the following theorem.

THEOREM 2.4 [3]. *Let X be a 1-connected space, $g : BV \to X$ a map, and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Assume that H^*X is of finite type, that $T^V_\gamma H^*X$ is of finite type, and that $T^V_\gamma H^*X$ vanishes in dimension 1. Then $\lambda_{X,g}$ is an isomorphism.*

For any object M of $\mathcal{U}$ the adjunction map $M \to H^V \otimes_{\mathbf{F}_p} T^V(M)$ can be combined with the unique algebra map $H^V \to \mathbf{F}_p$ to give a map $M \to T^V(M)$; call this map ϵ. (If $M = H^*X$ for some space X, then ϵ fits into a commutative diagram involving λ_X and the cohomology homomorphism induced by the basepoint evaluation map $\operatorname{Map}(BV, X) \to X$.)

THEOREM 2.5 [4, 6.3.2]. *The map $\epsilon : M \to T^V(M)$ is an isomorphism iff M is locally finite as a module over $\mathbf{A}_p$.*

If $R \in \mathcal{K}$, $M \in \mathcal{U}(R)$ and $\gamma : R \to H^V$ is a $\mathcal{K}$-map, we will denote the composite $M \xrightarrow{\epsilon} T^V(M) \to T^V_\gamma(M)$ by ϵ_γ. Theorem 2.5 leads to the following result, which we will need in §4.

PROPOSITION 2.6. *Let M be an object of $\mathcal{U}(H^V)$ and $\iota : H^V \to H^V$ the identity map. Then $\epsilon_\iota : M \to T^V_\iota(M)$ is an isomorphism iff M splits as a tensor product $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$ which is locally finite as a module over $\mathbf{A}_p$.*

PROOF: The fact that ϵ_ι is an isomorphism if M has the stated tensor product decompositon follows directly from 2.3(2), 2.5 and [**2**, 4.2]. Conversely, under the assumption that ϵ_ι is an isomorphism Proposition 2.4 of [**1**] guarantees that M splits as a tensor product $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$; the fact that N is locally finite is again a consequence of 2.3(2) and 2.5.