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COMPUTATION OF THE TOPOLOGY OF A REAL CURVE

M.-F. ROY

The problem of giving an algorithm for the computation of the topological type of a real curve from its equation has been considered by several authors ([GT] and more recently in the particular case of non singular curves by [AM].) The approach presented here relies on a basic result in real algebraic geometry, Thom's lemma; it can be viewed as an illustration of the philosophy in [CR]. Our algorithm runs in polynomial-time, needs no regularity hypothesis on the curve or on the projection and seems better adapted to situations where the connected components of the curve are small.

1. TERMINOLOGY AND GENERAL DISCUSSION

Let us introduce first some definitions.

1. 1. SEMI-ALGEBRAIC SETS

Let $F=(P_1(X_1,\dots,X_n),\dots,P_m(X_1,\dots,X_n))$ be a family of polynomials with integer coefficients. An *F-semi-algebraic set* S of $\mathbb{R}^n$ is a semi-algebraic set contained in $\mathbb{R}^n$, described by a boolean combination of sign conditions on the polynomials of F. A *F-basic semi-algebraic set* S of $\mathbb{R}^n$ is a semi-algebraic set contained in $\mathbb{R}^n$, described by a conjunction of sign conditions on the polynomials of F. One can know from basic results of real algebraic geometry ([B C R]) that a set S is semi-algebraic (because it is described by a first order formula of the language of ordered fields, using Tarski-Seidenberg, or because it is a connected component of a semi-algebraic set) without knowing polynomials F such that S is F-semi-algebraic.

1. 2. CYLINDRICAL DECOMPOSITIONS AND STRATIFICATIONS

Depending on the problem one wants to solve about semi-algebraic sets, one can use several kinds of cylindrical decompositions ([C], [Co]).

Let S be a F-semi-algebraic set, Π the canonical projection of $\mathbb{R}^n$ on $\mathbb{R}^{n-1}$.

(D_1) A *cylindrical decomposition* of F with respect to Π is given by a partition of $\mathbb{R}^{n-1}$ in a finite number of semi-algebraic sets A_i , such that above each A_i

(a) the real roots of the non-identically nul P_i are in constant number and given by continuous semi-algebraic functions $\zeta_{i,1} < \ldots < \zeta_{i,\, l_i}$;
(b) for all $x = (x_1,\ldots,x_{n-1})$ of A_i and all $j = 0,\ldots, l_i$ the sign of $P_i\,(x_1,\ldots,x_{n-1}, X_n)$ between $\zeta_{i,\,j}\,(x)$ and $\zeta_{i,\,j+1}(x)$ (with the convention $\zeta_{i,0}(x) = -\infty$ and $\zeta_{i,\,l_i+1}(x) = +\infty$) is fixed.

It is then clear that S is a finite union of *cells of the cylindrical decomposition* that is of graphs of $\zeta_{i,j}$, of slices between two $\zeta_{i,j}$ and $\zeta_{i,j+1}$ and of cylinders of the form $A_i \times \mathbb{R}$.

Let us remark that we do not ask for the A_i to be connected. If they are connected (b) is a consequence of (a).

(D_2) An *explicit cylindrical decomposition* of F is given by a family of polynomials F' (containing the polynomials F) and a cylindrical decomposition of F such that the sets A_i and the graphs of the $\zeta_{i,j}$ are F'-semi-algebraic sets.

Cylindrical decompositions give no information on adjacency relations: is A_i contained in the closure of $A_{i'}$? if it is the case, how do the functions $\zeta_{i',j}$ glue above A_i i.e. in which case is the graph of $\zeta_{i,j}$ contained in the closure of the graph of $\zeta_{i',j}$?

From now on, we shall suppose the polynomials of F monic with respect to X_n (which means that considered as polynomials in X_n, their leading coefficient belongs to $\mathbb{R}$). This can always be done by a linear change of coordinates.

(D_3) A *semi-algebraic stratification of* F is given by a cylindrical decomposition of F such that
(a') for all i, A_i is connected
(b') for all i and i' (i) either $A_i \cap A_{i'}$ is empty
(ii) or $A_i \subset \mathrm{adh}(A_{i'})$
(iii) or $A_{i'} \subset \mathrm{adh}(A_i)$
and given a pair (i, i') one explicitely knows which is the case
(c') one knows in case (ii) the adjacency relations between the $\zeta_{i,j}$ and the $\zeta_{i',\,j}$, i.e. for each pair (i, i') such that (ii) and for all (j, j') with $j \in \{1,\ldots,l_i\}$, $j' \in \{1,\ldots, l_{i'}\}$ one explicitely knows whether the graph of $\zeta_{i,j}$ is contained in the closure of the graph of $\zeta_{i',j'}$ or not.

(D_4) An *explicit semi-algebraic stratification* of F is a semi-algebraic stratification which is an explicit cylindrical decomposition.

1. 3. DIFFERENT SORT OF PROBLEMS

Let us consider now the following problems (S is a semi-algebraic set contained in $\mathbb{R}^n$:

(P_1) is S empty?

(P_2) what is the dimension of S?

(P_3) what is the number of connected components of S?

(P_4) what are the topological invariants of S (homology groups)?

(P_5) do two points of S belong to the same connected component of S?

(P_6) does a point belong to the projection of S?

(P_7) what is the explicit semi-algebraic description of the projection of S on $\mathbb{R}^{n-1}$?

(P_8) what is the explicit semi-algebraic description of $\{x \in \mathbb{R}^n \mid \Phi(x)\}$, where Φ is a first order formula of the language of ordered fields?

For the problems (P_1) to (P_5), one can choose the projection, that is one can make a linear change of coordinates, not for the problems (P_6) to (P_8).

A cylindrical decomposition allows to answer to (P_1), (P_2) and (P_6). An explicit cylindrical decomposition also to (P_7) and (P_8) (by induction). A semi-algebraic stratification to (P_3), (P_4) and (P_5). An explicit semi-algebraic stratification allows, in case the polynomials are monic with respect to the last variable (since in the problem (P_6) to (P_8) one cannot change the direction of the projection), to obtain (P_1) to (P_8).

A typical problem of robotics "à la Schwartz and Scharir ", the piano's mover problem [SS] is naturally formulated in term of (P_8) and (P_5) : one asks whether, semi-algebraic walls being explicitely given, there exists a movement allowing to pass from a position of an objet (the piano) to another without knocking the walls; one then considers the explicit semi-algebraic set S of allowed positions (problem (P_8)) and one answers yes if the initial and final positions belong to the same connected component of S.

1. 4. DIFFERENT FAMILIES OF POLYNOMIALS

It is not surprising that the computations to obtain these different sorts of cylindrical decompositions are different.

Different families of polynomials are to consider.

(F_1) A family of polynomials $F'=(P_{k,j}(X_1,\ldots,X_k)\ k=1,\ldots,n,\ j=1,\ldots,r_k)$ *is cylindrifying* for F if it contains F and is stable for the following operations:

-if $P(X_1,\dots,X_k)$ and $Q(X_1,\dots,X_k)$ are in F' the leading term of P, Q and of the subresultants of P and Q with respect to X_k are in F'
-if $P(X_1,\dots,X_k)$ is in F' the leading terms of the subdiscriminants of P with respect to X_k are in F'.

A family of polynomials $P_{1,j}(X_1)$, $j=1,\dots,l_1$, is always cylindrifying; the cells of the cylindrifying family are the real roots of the $P_{1,j}(X_1)$, $j=1,\dots,l_1$ and the intervals between such roots.

(F_2) A family of polynomials $P_{k,j}(X_1,\dots,X_k)$ $k=1,\dots,n$, $j=1,\dots,r_k$ is *glueing* for F if it contains a cylindrifying family for F, $Q_{k,j}(X_1,\dots,X_k)$ $k=1,\dots,n$, $j=1,\dots,r'_k$, and all the derivatives of the $Q_{k,j}(X_1,\dots,X_k)$ with respect to X_k.

(F_3) A family of polynomials $P_{k,j}(X_1,\dots,X_k)$ $k=1,\dots,n$, $j=1,\dots,r_k$ is *stratifying* for F if it is cylindrifying and stable under derivation (i.e. the family contains the derivative with respect to the variable X_k of the polynomials $P_{k,j}$).

Let us consider a real plane curve C of equation $P(X,Y)=0$, with P monic as polynomial in Y, squarefree and with coefficients in $\mathbb{Z}$, let D be the discriminant of P with respect to the variable Y.

Let us precise in this simple situation what are the different families of polynomials we have just defined:

(F_1) A cylindrifying family for P consists of P and D the discriminant of P with respect to Y.

(F_2) A glueing family for P consists of P and its derivatives with respect to Y, as well as D and its derivatives with respect to X.

(F_3) A stratifying family for P consists of P and its derivatives with respect to Y, of the discriminant D and the resultants obtained by eliminating Y between the different derivatives of P with respect to Y, then of the derivatives in X of D and of these resultants.

A cylindrifying family gives (D_1), a stratifying family (D_2) (in this case the cells of the cylindrical decomposition are basic semi-algebraic sets). If $P_1(X_1,\dots,X_n)$,..., $P_m(X_1,\dots,X_n)$ are monic with respect to X_n a glueing family gives (D_3) and a stratifying family (D_4).

The passage from (F_1) to (D_1) is done by Collins [C]: one takes the polynomials in X_1 belonging to the cylindrifying family, one isolates their