Astérisque

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Astérisque, tome 198-199-200 (1991), p. 153-163
<http://www.numdam.org/item?id=AST_1991__198-199-200__153_0>

THE NUMBER OF ABELIAN GROUPS OF ORDER AT MOST x

by

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1. Introduction

Let $a(n)$ denote the number of isomorphism classes of Abelian groups of order n. The arithmetic function $a(n)$ is multiplicative, and has a generating series

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\cdots \quad .$$

We shall be concerned here with the counting function

$$A(x) = \sum_{n \le x} a(n) \ ,$$

first considered by ERDŐS and SZEKERES [2]. One expects that $A(x)$ will be approximated by $\sum c_j x^{1/j}$, where

$$c_j = \prod_{\substack{k=1 \\ k \ne j}}^{\infty} \zeta(\frac{k}{j}).$$

Indeed, if we write

$$A(x) = \sum_{j=1}^{5} c_j x^{1/j} + \Delta(x), \tag{1.1}$$

then it is known on the one hand that

$$\Delta(x) \ll x^{97/381}(\log x)^{35}$$

(KOLESNIK [8]), and on the other, that

$$\int_1^X \Delta(x)^2 dx = \Omega(X^{4/3}\log X) \tag{1.2}$$

(IVIĆ **[7]** ; see also BALASUBRAMANIAN and RAMACHANDRA **[1]**). Thus

$$\Delta(x) = \Omega(x^{1/6}(\log x)^{1/2}),$$

so that the extra terms in the sum (1.1) that would correspond to $j \geq 6$, cannot be relevent. Note that

$$\frac{97}{381} = 0.25459\cdots > 0.16666\cdots = \frac{1}{6}\cdot$$

Our aim is to prove an upper bound corresponding to (1.2).

THEOREM 1. *We have*

$$\int_1^X \Delta(x)^2 dx \ll X^{4/3}(\log X)^{89}$$

for $X \geq 2$.

Apart from the exponent of $\log X$ this is, of course, best possible. IVIĆ **[6]** has given a weaker estimate with exponent $\frac{39}{29}$ in place of $\frac{4}{3}$. A result similar to Theorem 1 was stated by BALASUBRAMANIAN and RAMACHANDRA **[1]**, but it appears that their claim cannot be substantiated. We have made no attempt to obtain a good exponent for the power of $\log X$ in Theorem 1.

Our method is an elaboration of that used by the author **[4]** to estimate

$$\int_0^T \left|\zeta(\frac{5}{8}+it)\right|^8 dt\ .$$

We take this opportunity to point out that exactly the same technique yields :

THEOREM 2. *We have*

$$\int_0^T \left|\zeta(\frac{11}{20}+it)\right|^{10} dt \ll T^{3/2}(\log T)^{52}$$

and

$$\int_0^T \left|\zeta(\frac{9}{20}+it)\right|^{10} dt \ll T^{2}(\log T)^{52}$$

for $T \geq 2$. *Hence, in the generalized divisor problem, one has* $\beta_5 \leq \frac{9}{20}\cdot$

These results (with the exponent 52 replaced by 50) have been given without proof by ZHANG **[11]**.

Finally we observe that our method for proving Theorem 1 has a little to spare. An examination of the proof shows that the key estimate (3.1) can be obtained with a saving of a power of T, except when M and N differ only by a factor of a small power of T. In this latter case further arguments are available covering all possibilities except that in which M and N are both small powers of T. This argument suggests that one might actually hope to obtain an asymptotic formula for the integral in (1.2).

2. Mean-Value Bounds

To estimate the average of $\Delta(x)^2$ we shall use the analysis of IVIĆ [**6**; pp.19-21]. After suitable modifications, this leads to

$$\int_{X/2}^{X} \Delta(x)^2 dx \ll X^{4/3}(\log X)^8 \max_{1\le T\le X} T^{-1} I_T \ , \tag{2.1}$$

where

$$I_T = \int_{T/2}^{T} |\zeta(1-\sigma+it)\zeta(1-2\sigma+2it)\zeta(3\sigma+3it)\zeta(4\sigma+4it)\zeta(5\sigma+5it)|^2 dt \ ,$$

and

$$\sigma = \frac{1}{6} + \frac{1}{\log X}.$$

In view of the inequality $2|ab| \le a^2+b^2$, we have

$$I_T \le \max(J_T, J_T') \ , \tag{2.2}$$

where

$$J_T = \int_{T/2}^{T} |\zeta(3\sigma+3it)^2\zeta(4\sigma+4it)^4\zeta(5\sigma+5it)^4| dt \tag{2.3}$$

and

$$J_T' = \int_{T/2}^{T} |\zeta(3\sigma+3it)^2\zeta(1-\sigma+it)^4\zeta(1-2\sigma+2it)^4| dt \ .$$

Since the estimation of J_T and J_T' is similar, we shall henceforth restrict our attention to J_T.

We replace the integral in (2.3) by a sum over well-spaced points $t_n \in [T/2, T]$ for which

$$|t_m - t_n| \ge 1 \quad (m \ne n) \ . \tag{2.4}$$

Since

$$\zeta(s) = \sum_{n\le K} n^{-s} + 0(1) \quad (T \le K \le 2T)$$

for

$$|\mathrm{Im}\,(s)| \le 5T \ , \quad \frac{1}{2} \le \mathrm{Re}\,(s) \le \frac{7}{8} \ ,$$

by TITCHMARSH [10, Theorem 4.11], we have, for example

$$\zeta(3\sigma + 3it) \ll (\log T) \max_{L \le T} |S_3(L, 3t)| \ , \tag{2.5}$$

where L runs over powers of 2, and

$$S_3(L, 3t) = \sum_{L < n \le 2L} n^{-3\sigma - 3it} \ .$$

Of course, for the value of L giving the maximum in (2.5) we will clearly have

$$|S_3(L, 3t)| \ge |S_3(1, 3t)| \gg 1 \ .$$

Similarly

$$\zeta(4\sigma + 4it) \ll (\log T) \max_{M \le T} M^{-1/6} |S_4(M, 4t)|$$

with

$$S_4(M, 4t) = \sum_{M < n \le 2M} M^{1/6} n^{-4\sigma - 4it} \ , \tag{2.6}$$

and

$$\zeta(5\sigma + 5it) \ll (\log T) \max_{N \le T} N^{-1/3} |S_5(N, 5t)| \ ,$$

with

$$S_5(N, 5t) = \sum_{N < n \le 2N} N^{1/3} n^{-5\sigma - 5it} \ . \tag{2.7}$$

It follows that

$$J_T \ll (\log T)^{13} M^{-2/3} N^{-4/3} \sum_n |S_3(L, 3t_n)^2 S_4(M, 4t_n)^4 S_5(N, 5t_n)^4|$$

for certain fixed L, M, N with

$$|S_3(L, 3t_n)|, |S_4(M, 4t_n)|, |S_5(N, 5t_n)| \gg 1 \ .$$

We proceed to classify the points t_n according to the ranges

$$U < |S_3| \le 2U \ , \quad V < |S_4| \le 2V$$