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# EXPONENTIAL SUMS AFTER BOMBIERI AND IWANIEC

by

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BOMBIERI and IWANIEC [BI1, BI2] obtained $\theta = 9/56$ for the Lindelöf exponent (the least $\theta$ for which the Riemann zeta function satisfies $\zeta(1/2+it) = O(t^{\theta+\epsilon})$ as $t \to \infty$.)

They remarked that their method might not be special to the Lindelöf problem ; in fact, as the saying goes, "they wrought [worked] better than they knew".

To show that one property is uniformly distributed with respect to another property, one forms exponential sums

$$S = \sum_{M}^{2M-1} e(f(m)) \ , \tag{1}$$

where

$$e(x) = \exp 2\pi i x, \qquad f(m) = TF(m/M)$$

with $F(x)$ in the function class $C^n[1-\delta, 2+\delta]$ for some $\delta > 0$ and $n \geq 4$. The case $F(x) = \log x$ gives Dirichlet series. If $F(x)$ is a polynomial of degree $d$ with rational coefficients, denominator $q$, and if $T = M^d$, then the sum $S$ is approximately

$$MS_q/q \ ,$$

where $S_q$ is a complete exponential sum with denominator $q$. One imposes conditions to prevent $F(x)$ from being well approximated by a polynomial for a long interval of values of $m$. A sufficient condition is that $F(x)$ be holomorphic on a neighbourhood of the segment $1 \leq x \leq 2$ of the real axis, and satisfies there

$$F'(x) = (1 + o(1))x^{-s}$$

for some real $s > 0$. This condition is called the "virial" or "monomial" condition. It holds in many applications.

There are three useful ideas for treating exponential sums :
O. Subdivide the range for $m$,
A. Cauchy's inequality,
B. Poisson summation.
The name "Step A" is usually given to Weyl's differencing lemma, which may be analysed as subdivision, followed by Cauchy, followed by averaging. Van der Corput's method [see GK, I or K]consists of iterating these steps. The simplest form of Van der Corput's method, applying steps O, A, B (read from left to right) gives

$$S = O(M^{1/2}T^{1/6}) .$$

The method can be applied to exponential sums in several variables, and it becomes extremely complicated.

Bombieri and Iwaniec obtained

$$S = O(M^{1/2}T^{9/56+\varepsilon})$$

by taking the steps in the order O, B, A. The method is arithmetic, and is essentially limited to one variable. Their subdivisions correspond to approximations to $f(m)$ by quadratic polynomials with rational coefficients. If the denominator $q$ of the leading coefficient is small, the short interval is a "major arc", length $N$ say, and the sum over the short interval is approximately

$$NS_q/q .$$

If $q$ is large, the short interval is a "minor arc", and one expects the sum over the short interval to be small. This behaviour is seen in computer studies of exponential sums, notably those of DESHOUILLERS [D]. The Cauchy inequality is employed to show that the minor arc contribution is small in $L_p$ norm (for some suitable $p$). In some ways the treatment resembles applying Hardy and Littlewood's Farey dissection to

$$\int_0^1 \sum_{n=1}^{N} e(f(n+\alpha M))d\alpha . \tag{2}$$

If all arcs are treated as major (steps O,B alone), one gets

$$S = O(M^{1/2}T^{1/6+\varepsilon}).$$

This method is no worse than that of Van der Corput.

At the same time JUTILA [J1-8] has been considering sums

$$\sum_{M}^{2M-1} \tau(m) e(f(m)) , \tag{3}$$

where $\tau(m)$ is the divisor function or the Fourier coefficient of a modular form, beginning with steps O, B where O is subdivision according to the rational approximation to the first derivative, B is Voronoi or Wilton summation. In this context the numbers $\tau(m)\, e(-am/q)$ are the coefficients of the modular form twisted by the matrix $\begin{bmatrix} q & -a \\ 0 & q \end{bmatrix}$, and the Wilton summation formula is still available. These ideas could extend to any motivic $L$-series characterised by the three conditions :

D. An ordinary Dirichlet series with denominators $n^{-s}$,
E. An Euler product,
F. Functional equations for the $L$-series and its twists.

One may fit Bombieri and Iwaniec's ideas into this frame by taking $\tau(m)$ to be the theta-function coefficients, 2 if $m$ is a perfect square, 0 if not, and by considering $F(x)$ as a function of $x^2$. This change of variable explains why the derivatives do not correspond.

There are two successful applications of the Bombieri-Iwaniec method to sums with an extra variable. The Weyl step O, A replaces the sum $S$ of (1) with double sums of the form

$$\sum_{h=H}^{2H-1} \sum_{m=M}^{2M-1} e(f(m+h) - f(m)). \tag{4}$$

This sum suggests the simpler sum

$$\sum_{h=H}^{2H-1} \sum_{m=M}^{2M-1} e(hf'(m)). \tag{5}$$

The sum (5) was estimated by IWANIEC and MOZZOCHI [IM] using the same method. The rational polynomial approximation to $hf'(x)$ is found by multiplying the approximation to $f'(x)$ by $h$, so $h$ must not be too large. HEATH-BROWN and HUXLEY [HBH] estimated (4) - actually in the form

$$\sum_{h=H}^{2H-1} \sum_{m=M}^{2M-1} e(f(m+h) - f(m-h)). \tag{6}$$

This in turn gives estimates for

$$\int_{-U}^{U} |S(T_o + T)|^2 dT, \tag{7}$$

where $S(T)$ is the sum (1) considered as a function of $T$, if $H$ goes up to $T_o/U$ in size.

More general multiple exponential sums have not been treated, since one cannot find a good approximation by a rational polynomial.

The Iwaniec-Mozzochi sum (5) is connected with numerical integration. The prettiest case is the discrepancy for a circle (or more generally a smooth closed curve), the number of integer points minus the area. For a circle radius $R$, approximating the circle by a polygon whose sides lie along lattice lines $x =$ integer, $y =$ integer shows that the discrepancy is $O(R)$. Voronoi's method, applied by Sierpiński, approximates the circle by a polygon with rational gradients. Sierpiński obtained a discrepancy $O(R^{2/3})$ if the centre of the circle is at an integer point. The method can be modified [H2] to give $O(R^{2/3}(\log R)^{4/3})$ in general.

Exponential sums are introduced by way of the row-of-teeth function

$$\rho(t) = [t] - t + 1/2 \;\cong\; \sum_{h \neq 0} \frac{e(ht)}{2\pi i h} \, .$$

Thus

$$\sum_{m} \rho(\sqrt{(R^2 - m^2)})$$

can be expressed in terms of terms of the sums (5). The subdivision in step O corresponds to the sides of the Sierpiński polygon, with $q$ as the denominator of the rational gradient $a/q$.

Minor arc contributions can be classified as follows.

E1. The "main term", estimated in $L_p$ norm,
E2. Edge effects from ends of ranges of summation,
E3. Approximation errors in each summand in each Poisson summation.

The O, B, A sequence is dangerous because the errors of types (E2) and (E3) from each short sum in the subdivision must be added. For the sum $S$ of (1) there is a finite Poisson summation modulo $q$, followed by a Poisson summation in $m$, giving an Airy integral. For the double sums (5) and (6)