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On the Many Body Problem in Quantum Mechanics

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Section 1. Introduction

The aim of these lectures is to describe some of the modern mathematical techniques of N–body Scattering and with particular mention of their relations to other fields of analysis.

Consider a system of N quantum particles moving in $\mathbb{R}^n$, interacting with each other via the pair potentials V_α; the Hamiltonian (with center of mass removed) for such a system is given by

$$H = -\Delta + \sum_{i<j} V_{ij}(x_i - x_j) \quad \text{on} \quad L^2(\mathbb{R}^{n\,N-n}) \ .$$

Here $1 \leq i,\, j \leq N$, $x_i \in \mathbb{R}^n$. $-\Delta$ is the Laplacian on $L^2(\mathbb{R}^{n\,N-n})$ with metric

$$x \cdot y = \sum_{i=1}^{N} m_i x_i \cdot y_i \qquad ; \quad m_i > 0 \ .$$

The m_i are the masses of the particles. The main problem of scattering theory is to describe the spectral properties of H and find the asymptotic behavior of $e^{-iHt}\varphi$ for $\varphi \in L^2$, as $t \to \pm\infty$.

There are two reasons for that: one, the behavior is much simpler as $t \to \pm\infty$. Secondly it determines the full properties of the system. Since the

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sum $\sum_{i<j} V_{ij}$ does not vanish as $|x| \to \infty$ in certain directions, the perturbation of $-\Delta$ is not negligible at infinity. The spectral properties and asymptotic behavior of H are therefore radically different than that of $-\Delta$.

This is the generic multichannel problem. There are many different asymptotic behaviors possible, depending on the choice of φ. Thus the main theorem can be phrased as: given $\varphi \in L^2(\mathbb{R}^n)$, find hamiltonians H_a and functions $\varphi_a^\pm$, s.t.

$$e^{-iHt}\varphi - \sum_a e^{-iH_a t}\varphi_a^\pm \approx 0 \qquad \text{as } t \to \pm\infty .$$

Accepting the physicist's dogma that every state of the system is described asymptotically in terms of *particles* (or bound clusters of particles) we conclude that the only possible H_a are the subhamiltonians of the system:

$$H_a = H - I_a$$
$$I_a \equiv \sum_{(i,j)\subset a} V_{ij}(x_i - x_j)$$

and a stands for arbitrary disjoint cluster decomposition of $\{1, 2, \dots, N\}$.

I_a is called the intercluster interaction. The Hamiltonian that describes the bound clusters of a decomposition a, is denoted by H^a. Not much is known for Multichannel Non Linear Scattering; see however [Sof-We and cited ref.].

The approach to studying $e^{-iHt}\psi$ for large $|t|$ is by first reducing the problem via channel decoupling (or other methods) to the study of the localization in the phase space of $e^{-iHt}\psi$. Then, we develop a theory of propagation in the phase space for H. The channel decoupling is achieved by constructing a partition of unity of the space, with two main properties: one, on the support of each member of the partition the motion $e^{-iHt}\psi$ is simple (= one channel) and can be described by one fixed hamiltonian. The second property is that the boundary of the partitions is localized in regions where we can prove that no propagation of $e^{-iHt}\psi$ is possible there for large times; in this way we conclude that no switching back and forth between channels is possible as $|t| \to \infty$ which implies the desired results.

The first part, based on the construction of partitions of unity relies mainly on geometric analysis combined with the kinematics of (freely) moving particles. Different techniques are now known, each with its own importance, and I will describe some of the main constructions. The second part of the proof is analytic; it provides an approach to finding the asymptotic behavior of $e^{-iHt}\psi$ as $|t| \to \infty$, which is *complementary to that of stationary phase.* As I will describe below it replaces the (central) notion of *oscillation* by that of *microlocal monotonicity.* The distinctive feature of this approach allows the study of general pseudo differential operators H on equal footing with constant coefficient operators.

The first proof of Asymptotic Completeness (AC) for N-body systems along these lines was given in [Sig-Sof1]. Since then, different proofs were developed, with new useful implications [Der1, Kit, Gr, Ta] (see also [En2, Ger2-3]). Further developments concentrated on the long range problem. The three body case was first solved by Enss [En2]. (See also [Sig-Sof3].) Local decay and minimal and maximal velocity bounds were proved for N-body hamiltonian, including ones with time dependent potentials in [Sig-Sof2]. This approach is further utilized in [Sk, FrL, Ger2 , Ger-Sig, H-Sk]. A method of dealing with the problem of AC for long range many body scattering is developed in [Sig-Sof4,5]; the case of $N = 4$ is solved there.

A final comment; the phase space approach to N-body scattering originated with the fundamental works of Enss [En1,2]. A comprehensive description of the Enss method can be found in [Pe], including applications to many problems in spectral theory. References of many of by now classical results, including the works of Mourre, until about 1983 can be found in [CFKS]. We refer the reader to this book also as the basic reference used here on spectral and scattering theory.

Section 2. Microlocal Propagation Theory

Let H be a self adjoint operator on $L^2(\mathbb{R}^n)$ arising from the quantization of a classical Hamiltonian h. By solving the Hamilton-Jacoby equations for h it makes sense to talk about the classical trajectories (or bi-characteristics) of h (or H). As $t \to \pm\infty$ the (unbounded) trajectories concentrate, in general, in a certain set of the phase space.

DEFINITION 2.1. A bounded p.d.o. j with symbol homogeneous of degree 0 in x is said to be supported away from the *propagation set* (at energy E) of H if the following estimate holds

$$\int_{\pm 1}^{\pm\infty} \| \frac{1}{\langle x\rangle^{1/2}} j\, e^{-iHt}\psi\|^2 dt \le c\|\psi\|^2 \quad \text{for all} \quad \psi = E_\Omega(H)\psi \ .$$

Here $\langle x\rangle^2 \equiv 1 + x^2$, $E_\Omega(H)$ is the spectral projection of H, with Ω any sufficiently small interval containing E.

Our aim is to identify the (conical) set PS_E of the phase space, with the property that any j is supported away from the propagation set in the sense of the above definition if and only if it is supported away from PS_E. We can therefore think of PS_E as the *propagation set of H at energy E.*

The main tool to proving that a given conical set $\tilde{K}$ is away from the propagation set PS_E will be to prove (microlocal) monotonicity of the flow generated by H in $\tilde{K}$.

The claim is that the *classical* flow generated by H is moving out of any such $\tilde{K}$ monotonically in t, for large t. By finding a lower bound for this monotone flow in $\tilde{K}$ we can then absorb the effects of quantization and other potential perturbations of H.

I chose to describe the above approach first when applied to $H = -\Delta$, and along the way prove some known and new smoothing estimates for $-\Delta$. The proofs are easy but allow the introduction of some of the other fundamental notions and arguments repeatedly used later.

DEFINITION 2.2. The Heisenberg derivative of an operator family $F(t)$, $DF(t)$, w.r.t. to H is defined by

$$DF(t) \equiv i[H, F] + \frac{\partial F}{\partial t} \ ;$$

DEFINITION 2.3. A bounded family of linear operators $F(t)$ on $L^2(\mathbb{R}^n)$ is called a *propagation observable* for H if its Heisenberg derivative is positive –