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0. Introduction

The main purpose of these lecture notes, which are a revised and expanded version of the survey paper [S-U V], is to give an overview of the mathematical developments in the last few years in inverse boundary value problems. In these problems one attempts to discover internal properties of a body by making measurements at the boundary. We concentrate mainly in the problem of determining the conductivity of a body from measurements of voltage potentials and corresponding current fluxes at the boundary. This problem which is often referred to as *Electrical Impedance Tomography* arose in geophysics from attempts to determine the composition of the earth. More recently it has been proposed as a potentially valuable diagnostic tool for the medical sciences. The methods developed to study this problem have lead to new results in inverse scattering and inverse spectral problems. We also give an account of some of these developments in these notes.

1. Electrical impedance tomography; the isotropic case.

In this section we formulate the inverse conductivity problem and a similar problem for the Schrödinger equation at zero energy.

Let $\Omega \subseteq \mathbf{R}^n$ $n \geq 2$, be a smooth bounded domain. If the conductivity of Ω is independent of direction (isotropic case) it is represented by a positive function, which we assume in $C^{1,1}(\overline{\Omega})$, with a positive lower bound. If we assume that there are no sources or sinks of current in Ω, the conductivity equation for the potential u in Ω is

$$L_\gamma u = \operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega. \tag{1.1}$$

If f represents the induced potential on the boundary (assume $f \in H^{\frac{1}{2}}(\partial\Omega)$), $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} L_\gamma u &= 0 \qquad \text{in } \Omega \\ u|_{\partial\Omega} &= f. \end{aligned} \tag{1.2}$$

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The Dirichlet to Neumann map is then defined by

$$\Lambda_\gamma(f) = \gamma\frac{\partial u}{\partial \nu} \tag{1.3}$$

where u is the solution of (1.2) and ν is the unit outer normal to the boundary. The map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is selfadjoint and is often called the voltage to current map because $\gamma\frac{\partial u}{\partial \nu}$ measures the current flux at the boundary.

The inverse conductivity problem consists of the study of various properties of the map

$$\gamma \xrightarrow{\Phi} \Lambda_\gamma. \tag{1.4}$$

These properties include the injectivity, range, and continuity of the map and its inverse (when an inverse exists). From the point of view of applications, an even more important problem is to give a method to reconstruct γ(or at least to deduce as much information as possible about γ) from Λ_γ.

A closely related problem is to consider instead of the conductivity equation, the Schrödinger equation at zero energy

$$L_q = \Delta - q \tag{1.5}$$

where $q \in L^\infty(\Omega)$.

If 0 is not an eigenvalue of L_q, we can solve the Dirichlet problem

$$\begin{aligned} L_q u &= 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} &= f \end{aligned} \tag{1.6}$$

and define the Dirichlet to Neumann map by

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu} \tag{1.7}$$

where u is the solution of (1.6). We want to study the map

$$q \xrightarrow{\widetilde{\Phi}} \Lambda_q. \tag{1.8}$$

Λ_γ and Λ_q are related in the following way: If u is a solution of (1.1) then

$$w = \gamma^{\frac{1}{2}} u$$

is a solution of $L_q w = 0$ with $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. It is a straightforward computation to see that

$$\Lambda_q = \gamma^{-\frac{1}{2}}\Lambda_\gamma\gamma^{-\frac{1}{2}} + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu}. \tag{1.9}$$

Thus if we know $\Lambda_\gamma, \gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}$ we can determine Λ_q. In the next section we shall see that Λ_γ determines $\gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}$, so that knowledge of Λ_γ determines Λ_q.

2. Results at the boundary

Kohn and Vogelius ([K-V, I]) proved that if $\gamma \in C^\infty(\overline{\Omega})$ one can determine $\frac{\partial^j\gamma}{\partial\nu^j}|_{\partial\Omega} \quad \forall j$.

Theorem 2.1. *Let $\gamma_i (i = 1, 2)$ be in $L^\infty(\Omega)$ with a positive lower bound. Let $x_0 \in \partial\Omega$ and let B be a neighborhood of x_0 relative to $\overline{\Omega}$. Suppose that*

$$\gamma_i \in C^\infty(B), \quad i = 1, 2$$

and

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \quad \forall f \in H^{\frac{1}{2}}(\partial\Omega) \qquad \text{with}$$

supp $f \subset B \cap \partial\Omega$, then

$$(\frac{\partial}{\partial x})^\alpha \gamma_1(x_0) = (\frac{\partial}{\partial x})^\alpha \gamma_2(x_0)$$

where

$$(\frac{\partial}{\partial x})^\alpha \text{ denotes } (\frac{\partial}{\partial x_1})^{\alpha_1} \dots (\frac{\partial}{\partial x_n})^{\alpha_n}.$$

Sketch of proof.

Kohn and Vogelius proved this result by cleverly choosing boundary data. We outline here a different approach taken in [S-U, I] which makes use of the fact that Λ_γ is a pseudodifferential operator of order 1. This means that, in local coordinates near $x_0 \in \partial\Omega$ which we denote by x', and for f supported near x_0,

$$\Lambda_\gamma f(x') = \int e^{ix'\cdot\xi'}\lambda_\gamma(x', \xi')\hat{f}(\xi')d\xi'. \tag{2.2}$$

$\lambda_\gamma(x', \xi')$ is the full symbol of Λ_γ and has an asymptotic expansion for large $|\xi'|$

$$\lambda_\gamma(x', \xi') \sim \sum_{j\leq 1} \lambda_\gamma^{(j)}(x', \xi') \tag{2.3}$$

with $\lambda_\gamma^{(j)}$ homogeneous of degree j in ξ'. We have $\lambda_\gamma^{(1)}(x',\xi') = \gamma|_{\partial\Omega}(x')|\xi'|$ and it was proven in [S-U, I] that $\lambda_\gamma^{(j)}(x',\xi')$ determines inductively $\frac{\partial^{j-1}}{\partial\nu^{j-1}}\gamma|_{\partial\Omega}$ (For a simpler proof of this see the paper [L-U] and also the sketch in section 9 of this paper.) □

The previous result implies the injectivity of Φ at real-analytic conductivities. Kohn and Vogelius extended this result further to cover piecewise real-analytic conductivities ([K-V, II]).

Sylvester and Uhlmann ([S-U I]) used the proof of Theorem 2.1 outlined above to give continuous dependence estimates at the boundary.

Theorem 2.4. *Let γ_i, $i = 1,2$ be in $L^\infty(\Omega)$ with a positive lower bound. Then*

(a)
$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}} \leq C\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}.$$

If γ_1, γ_2 are continuous, then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C_1\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}.$$

(b) If γ_1, γ_2 are Lipschitz continuous then

$$B_i = \Lambda_{\gamma_i} - \gamma_i\Lambda_1 \qquad \textit{satisfy}$$

$$\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} \leq C_2\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\Omega)}$$

and $\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\partial\Omega)} + \|\frac{\partial}{\partial\nu}(\gamma_1 - \gamma_2)\|_{L^\infty(\partial\Omega)}$

$$\leq C_3(\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}})$$

On the operators we use the operator norm. C_1 depends only on Ω and the lower bound of the γ_i's. C_2, C_3 depends only on Ω and the γ_i's are normalized to have Lipschitz norm less than or equal to one.

3. Linearization at constants; Calderón's approach

Calderón formulated the inverse conductivity problem in a different way. He considered the Dirichlet integral associated to the solution of (1.2)

$$Q_\gamma(f) = \int_\Omega \gamma|\nabla u|^2, \tag{3.1}$$

$Q_\gamma(f)$ measures the power necessary to maintain the potential f on the boundary.