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SPECTRAL THEORY OF ELLIPTIC OPERATORS ON NON-COMPACT MANIFOLDS.

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Introduction

This paper contains an enlarged and modified part of my five lectures given in June 1991 at Nantes during the Summer School on Semiclassical Methods. Of course the whole subject as given in the title is inexhaustible since even the "simplest" particular case of the Schrödinger operator on euclidean space can not be exhausted because it contains the whole Quantum Mechanics and hence its complete understanding would provide us with the complete understanding of a considerable part of the Universe. So I did not pretend to be complete in my lectures and I make even less pretensions in this paper. Actually this paper contains only a description of some qualitative results on the spectra of elliptic operators on non–compact manifolds. The lectures contained also a beginning of a quantitative theory, namely integrated density of states and applications of von Neumann algebra techniques to this topic. I hope that these things some day will be described in a second part of this paper but they seemed to me too voluminous and disorderly to include in this paper now.

This paper contains two chapters each having an Appendix. In Chapter 1 we discuss the first question which natually arises when you begin to study a differential operator: what is the natural domain, where this operator is defined? Actually, if the operator is to be considered in a Banach space, one can always take minimal and maximal domain arriving in this way to minimal and maximal operators in this Banach space. We concentrate on the question whether these operators coincide because then they provide a natural operator in the Banach space associated with the given differential operator. We describe several methods of proving the coincidence based on finite speed propagation for evolution equations, regularity results and estimates of the Green function. The necessary technique concerning manifolds of bounded geometry and behaviour of the Green function is described in Appendix 1 to this chapter. Note that a non–trivial difference between minimal and maximal operator would mean that boundary conditions should be imposed but this certainly goes out of the scope of this paper. The only thing we do about it here is that we explain how to write the unique solution of the hyperbolic Cauchy problem in operator terms in case when the corresponding generating second order operator is symmetric but not essentially self–adjoint due to the behaviour of lower–order terms at infinity (Theorem 3.4).

In Chapter 2 we discuss some general topics concerning elliptic operators on manifolds of bounded geometry. Namely first we apply the general abstract eigenfunction expansion theorem, described in Appendix 2, to provide weighted Sobolev spaces which contain complete orthonormal system of generalized eigenfunctions for any self–adjoint operator. We use the ellipticity to narrow these spaces by use of regularity theorems. Next we discuss Schnol–type theorems giving sufficient conditions for the given complex number λ to belong to the spectrum if a non–trivial and non–square –integrable eigenfunction with an appropriate behaviour at infinity is given.

Some parts of this paper are based on methods and technique that were described in [44] and [45], and I felt free to borrow from these papers which were only published in a volume of the PDE seminar in École Polytechnique. But many of the results of [44] are essentially improved here and also some clarifications are added.

We are very grateful to the organizers of the Summer School on Semiclassical Methods at Nantes (and especially to Professor D. Robert) for providing the opportunity to lecture there and so to see the topics discussed here from a renewed point of view. We are also very grateful to the Sloan Foundation and M.I.T. for their support during the time when this text was being written, and to