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SEMICLASSICAL SPECTRAL ASYMPTOTICS

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0. Introduction

The problem of the spectral asymptotics, in particular the problem of asymptotic distribution of eigenvalues is one of the central problems of the spectral theory of partial differential operators. It is also very important for the general theory of partial differential operators. Apart from applications in the quantum mechanics, radiophysics, continuum media mechanics (elasticity, hydrodynamics, theory of shells) etc, there are also applications to the mathematics itself and moreover there are deep though non-obvious links with differential geometry, dynamic systems theory and ergodic theory; even the term "spectral geometry" has arisen. All these circumstances make this topic very attractive for a mathematician.

This problem originated in 1911 when H.Weyl published a paper devoted to eigenvalue asymptotics for the Laplace operator in a bounded domain with a regular boundary. After this article there was published a huge number of papers devoted to the spectral asymptotics and numerous prominent mathematicians were among their authors. The theory was developed in two directions: first of all this theory was extended and there were considered more and more general operators and boundary conditions as well as geometrical domains on which these operators were given; on the other hand the theory was improved and more and more accurate remainder estimates were derived. Namely in the later way the links with differential geometry, dynamic systems theory and ergodic theory appeared. Even the theory of eigenvalue asymptotics for the Laplace (or Laplace-Beltrami) operator has a long, dramatic and yet non-finished history. At a certain moment apart of asymptotics with respect to the spectral parameter there appeared asymptotics with respect to other parameters; the most important among them are (in my opinion) semiclassical asymptotics, i.e. asymptotics with respect to the small parameter h (Planck constant in physics) tending to $+0$. For a long time these asymptotics were in the shadow: most attention was paid to the eigenvalue asymptotics for operators on compact manifolds (with or without a boundary); the results which had been obtained here then were proved again for operators in $\mathbb{R}^d$ such as the Schrödinger operator $-h^2\Delta + V(x)$ with fixed $h > 0$ and with $V(x) \to +\infty$ as $|x| \to \infty$; less attention was paid to semiclassical asymptotics (i.e. asymptotics of eigenvalues less than some fixed level λ as $h \to +0$); moreover the asymptotics of the small negative eigenvalues were considered in the case of fixed h and $V(x)$ decreasing at infinity as $|x|^{2m}$ with $m \in (-1, 0)$; under reasonable conditions in this case the discrete spectrum of an operator has an accumulation point -0 and the essential spectrum coincides with $[0, +\infty)$. The result of the development of the theory described above was that at a certain moment there existed four parallel (though not equally developed) theories and the statements in each of them had to be proved separately. However now this plurality has been finished (at least in my papers) because all the other results are easily derived from the *local semiclassical spectral asymptotics* (*LSSA* in what follows), which are the main object of these lectures All other results are obtained as their applications.

In his papers H.Weyl applied the *variational* method (Dirichlet-Neumann bracketing) invented by himself; later this method was improved in various directions by many mathematicians. Other methods also appeared later and I would like to mention only the *method of a hyperbolic operator* due to B.M.Levitan and Avvakumovič[1)]. All the asymptotics with the most accurate remainder estimates were obtained by this method. It is based on the fact that the fundamental solution to the Cauchy problem (or the initial-boundary value problem) $u(x, x, t)$ for the operator $D_t - A$ is the Schwartz' kernel of the operator $\exp itA$ (where $D_t = -i\partial_t$, etc) and it is connected with the eigenvalue counting function of an operator A by the formula

$$\int u(x, x, t)\,dx = \int e^{it\lambda}\, d_\lambda N(\lambda); \tag{0.1}$$

in the case of a matrix operator A $u(x, y, t)$ is a matrix-valued function and in the left-hand expression it should be replaced by its trace. Here and below $N(\lambda)$ is the number of eigenvalues of A less than λ (and in this place we consider only operators semi-bounded from below with purely discrete spectra). Then by means of the inverse Fourier transform we can recover $N(\lambda)$ provided we have constructed $u(x, y, t)$ by means of the methods of theory of partial differential operators. However, in fact we are never able (excluding some very special

[1)] This method is a special case of Tauberian methods due to T.Carleman; resolvent method, method of complex power and method of heat equation are other Tauberian methods. The method of the almost spectral projector due to M.Shubin and V.Tulovskii lies between variational and Tauberian methods.

and rare cases when all this machinery is not necessary) to construct $u(x,y,t)$ precisely and for all the values $t \in \mathbb{R}$. Usually (now we assume that A is an elliptic first-order pseudo-differential operator) the fundamental solution is constructed approximately (modulo smooth functions) for t belonging to some interval $[-T,T]$ with $T > 0$. As a consequence we obtain modulo $O(\lambda^{-K})$ with any arbitrarily chosen K an expression for

$$F_{t\to\tau}\chi_T(t)\int u(x,x,t)dx = \int \hat{\chi}_T(\tau-\lambda)d_\lambda N(\lambda) \tag{0.2}$$

where χ is a fixed smooth function supported in $[-1,1]$, $\chi_T(t) = \chi(\frac{t}{T})$ and a hat as well as $F_{t\to\tau}$ mean the Fourier transform. Then if we know the left-hand expression, using the Tauberian theorem due to Hörmander we are able to recover approximately $N(\lambda)$ by the formula

$$N(\lambda) = \int_{-\infty}^{\lambda} (F_{t\to\tau}\chi_T(t)\sigma)(\tau))d\tau + O(\lambda^{d-1}) \tag{0.3}$$

where d is the dimension of the domain,

$$\sigma(t) = \int u(x,x,t)dx \tag{0.4}$$

and the explicit construction of $u(x,x,t)$ in this situation yields the formula

$$N(\lambda) = c_0\lambda^d + O(\lambda^{d-1}) \tag{0.5}$$

with the leading coefficient

$$c_0 = (2\pi)^{-d}\int_{a(x,\xi)<1} dxd\xi, \tag{0.6}$$

where $a(x,\xi)$ is a principal symbol of A.

We see that the crucial step in this approach is the construction of the fundamental solution. This construction by means of Fourier integral operators[2)] is standard and well-known now, provided we consider a scalar operator for an operator with constant multiplicities of the eigenvalues of the principal symbol and we construct $u(x,y,t)$ at the compact K contained in the interior of our domain X (and T depends on the distance between K and ∂X). If one of these assumptions is violated then the construction is more sophisticated and possible only under some very restrictive conditions. In the presence of a boundary (but only in the case of the constant multiplicities of the eigenvalues of the principal symbol) this construction was realized in certain papers due to R.Seeley, D.Vasil'ev, R.Melrose. However, it is possible to avoid all the troubles by means of another approach suggested by V.Ivrii[4] (see also L.Hörmander [3]) based on the investigation of the propagation of singularities

[2)] This construction due to L.Hörmander played a very important and stimulating role in the development of Fourier integral operators theory.

for $u(x, y, t)$ and construction of an "approximation" (in a rather exotic sense) for this distribution leading to an approximation in the reasonable sense for $\sigma(t)$ for $|t| \le T$ with appropriate $T > 0$. For h-pseudo-differential operators which are the main subject of this article this approach is essentially more simple and transparent because there is a selected parameter h. We'll discuss this case below. We'll be able to prove in this way the asymptotics (0.3) for an arbitrary self-adjoint m-th order elliptic operator with $m > 0$ and the spectral parameter λ^m now on a compact manifold without or with a boundary (in the former case the boundary conditions are also supposed to be elliptic), scalar or matrix, semi-bounded from below or non-semi-bounded at all (in this case $N(\lambda)$ is replaced by $N^{\pm}(\lambda)$ which is a number of eigenvalues lying between 0 and $\pm\lambda^m$); the formula for c_0 should be changed if it is necessary.

At the same time the two-terms asymptotics

$$N(\lambda) = c_0\lambda^d + c_1\lambda^{d-1} + o(\lambda^{d-1}) \tag{0.7}$$

suggested by H.Weyl (who also gave a formula for c_1) fails to be true unless some additional condition is fulfilled. It is certainly wrong for $d = 1$ and for the Laplace-Beltrami operator on the sphere $\mathbb{S}^d$ of any dimension (this is due to the high multiplicities of its eigenvalues). Moreover, this asymptotics remains wrong in the case when this Laplace-Beltrami operator is perturbed by a potential or even by a symmetric first-order operator with small coefficients; in this case all the eigenvalues of high multiplicities will generate narrow eigenvalue clusters separated by lacunae. On the other hand under some conditions of the global nature the asymptotics (0.7) is valid. For a scalar operator on a compact manifold without a boundary this condition is "*The measure of the* {*set of all the points of the cotangent bundle periodic with respect to the Hamiltonian flow generated by the principal symbol* } *equals to 0*"[3]. This condition is more complicated for matrix operators. For a scalar second-order operator on a compact manifold with a boundary one needs to consider only trajectories transversal to the boundary and reflecting according to the geometrical optics law. Though there are some points of the cotangent bundle through which such infinitely long trajectory doesn't pass, but the measure of these dead-end points vanishes and we do not have to take them into account. For higher-order operators as well as for matrix operators the trajectories reflected from the boundary can branch and in this case it is necessary to follow every branch. This makes the situation much more complicated and the following additional condition (which isn't automatically fulfilled now) appears "*the measure of the* {*set of all the dead-end points*} *equals to* 0".

Let us clarify for the scalar first-order operator on a manifold without boundary a link between asymptotics (0.7) and periodic Hamiltonian trajec-

[3]This condition appeared first in the papers of J.J.Duistermaat and V.Guillemin.