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DIVISOR FUNCTIONS OF INTEGER MATRICES:
EVALUATIONS, AVERAGE ORDERS AND APPLICATIONS

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Divisor functions have been extensively studied in classical number theory. We examine the corresponding functions for matrices over the ring of integers. These functions not only are a natural generalisation of historically important concepts, but their study also yields applications and some new interpretations.

We deal only with $r \times r$ nonsingular matrices with entries from the ring of integers. Since there are an infinite number of unimodular matrices, it is necessary to identify *canonical factorisations* of matrices. In his study of arithmetic of matrices, Nanda [8] defines them as follows:

DEFINITION 1. The decomposition of the matrix A as

$$A = A_1 A_2 \dots A_k, \tag{1}$$

where $A_k, A_{k-1}, \dots, A_2$ are matrices in nonsingular Hermite Normal Form (HNF), is said to be a k-order (canonical) factorisation of A.

Since an HNF matrix uniquely represents a class under one-sided equivalence, two factorisations defined as in (1) are *inequivalent*.

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DEFINITION 2. The function $\tau_k^{(r)}$ counts the number of k-order factorisations of an $r \times r$ matrix A, i.e.,

$$\tau_k^{(r)}(A) = \sum_{A=A_1 A_2 \ldots A_k} 1. \tag{2}$$

When we restrict ourselves to the case $k = 2$, A_2 above is called a *divisor* of A. As in classical theory, we define the functions $\sigma_a^{(r)}$.

DEFINITION 3. For a non-negative integer a, let $\sigma_a^{(r)}(A)$ be the sum of the a^{th} powers of the determinants of divisors of a matrix A, i.e.,

$$\sigma_a^{(r)}(A) = \sum_{A=A_1 A_2} (\det A_2)^a. \tag{3}$$

We notice that $\tau_2^{(r)} = \sigma_0^{(r)}$.

Evaluations

For evaluating divisor functions, it is enough to consider A to be in non-singular Smith Normal Form (SNF) with prime-power determinant. We use the notation F_r for the matrix

$$\left(\operatorname{diag}\left[p^{f_1}, p^{f_1+f_2}, \ldots, p^{f_1+f_2+\ldots+f_r}\right]\right)$$

and may also denote it by $\langle f_1, f_2, \ldots, f_r \rangle$, where $f_1 \geq 1,\ f_i \geq 0$ for $i > 1$, and p is a prime number. We write G_r for the matrix $\langle f_1, f_2, \ldots, f_{r-1}, f_r - 1 \rangle$.

It is clear that the evaluation of $\sigma_a^{(r)}$ involves the solving of a system of diophantine equations with rather stringent side conditions and is therefore not easy.

In [4] we proved the result:

$$\tau_2(F_r) - \tau_2(G_r) = \sigma_1(F_{r-1}), \tag{4}$$

which helped us evaluate $\tau_2^{(2)}$ and $\tau_2^{(3)}$. Here we generalise (4) and prove a recurrence between $\sigma_a^{(r)}$ and $\sigma_{a+1}^{(r-1)}$. Thus it becomes possible to reduce the problem to evaluations on F_1 or on *prime matrices* $P_r = \langle 1, 0, \ldots, 0 \rangle$. We prove:

THEOREM 1. $\sigma_a(F_r) - p^a\sigma_a(G_r) = \sigma_{a+1}(F_{r-1})$.

Proof. Let $\nu = f_1 + f_2 + \ldots + f_r$. Now

$$F_r = \begin{pmatrix} A_{r-1} & 0 \\ Y & p^t \end{pmatrix} \begin{pmatrix} B_{r-1} & 0 \\ X & p^{\nu-t} \end{pmatrix}, \qquad 0 \le t \le \nu \tag{5}$$

$$G_r = \begin{pmatrix} A_{r-1} & 0 \\ Y & p^t \end{pmatrix} \begin{pmatrix} B_{r-1} & 0 \\ X & p^{\nu-t-1} \end{pmatrix}, \qquad 0 \le t < \nu \tag{6}$$

are factorisations of F_r and G_r whenever B_{r-1} is a divisor of F_{r-1} and $YB_{r-1} + Xp^t = 0$. Then

$$\sigma_a(F_r) = \sum_{\substack{YB_{r-1}+Xp^t=0 \\ 0\le t<\nu}} (\det B_{r-1})^a\, p^{(\nu-t)a} + \sum_{YB_{r-1}+Xp^\nu=0} (\det B_{r-1})^a \tag{7}$$

and

$$\sigma_a(G_r) = \sum_{\substack{YB_{r-1}+Xp^t=0 \\ 0\le t<\nu}} (\det B_{r-1})^a\, p^{(\nu-t-1)a} \tag{8}$$

Thus

$$\sigma_a(F_r) = p^a\sigma_a(G_r) + \sum_{YB_{r-1}+Xp^\nu=0} (\det B_{r-1})^a. \tag{9}$$

But, as in the proof of (4), we can show that the equation $YB_{r-1} + Xp^\nu = 0$ has solutions for all possible choices of X by considering the cases where the matrix

$$\begin{pmatrix} B_{r-1}^{-1} F_{r-1} & 0 \\ -p^\nu X B_{r-1}^{-1} & p^\nu \end{pmatrix}$$

is integral. Since X can be chosen in $\det(B_{r-1})$ ways, we have:

$$\sigma_a(F_r) = p^a\sigma_a(G_r) + \sum_{B_{r-1}\,|\,F_{r-1}} (\det B_{r-1})^{a+1}.$$

□

It is easy to evaluate $\sigma_a(P_r)$ by generalising Nanda's result [7] for $a = 0$:

THEOREM 2. $\sigma_a(P_r) = \sum_{j=0}^{r} p^{aj} \begin{bmatrix} r \\ j \end{bmatrix}$, *where* $\begin{bmatrix} r \\ j \end{bmatrix}$ *are the Gaussian polynomials in* p.

Proof. Nanda [7] gives a combinatorial argument to show that the number of divisors of P_r with p occurring exactly j times on the diagonal is given by $\begin{bmatrix} r \\ j \end{bmatrix}$. The factor p^{aj} is obviously the a^{th} power of the determinant of the divisor. □

The above theorems give us a method for evaluation, but the calculations involved are cumbersome, as the following example shows.

EXAMPLE.

$$\begin{aligned}\sigma_1(F_3) = &\left(p^{f_3} + p^{f_3-1} + \ldots + 1\right)\sigma_2(F_2) \\ &+ p^{f_3+1}(p+1)\left(\sigma_2\langle f_1, f_2-1\rangle + p^2\sigma_2\langle f_1, f_2-2\rangle + \ldots + p^{2f_2-2}\sigma_2\langle f_1, 0\rangle\right) \\ &+ p^{f_3+2f_2+1}\sigma_2\langle f_1-1, 1\rangle + p^{f_3+2f_2+2}(p+1)\sigma_2\langle f_1-1, 0\rangle \\ &+ p^{f_3+2f_2+4}\sigma_2\langle f_1-2, 1\rangle + p^{f_3+2f_2+5}(p+1)\sigma_2\langle f_1-2, 0\rangle + \ldots \\ &+ p^{f_3+2f_2+3f_1+2}(p+1)\sigma_2\langle 1, 0\rangle + p^{f_3+2f_2+3f_1+4}\sigma_1\langle 1, 0\rangle,\end{aligned}$$

where

$$\begin{aligned}\sigma_2(F_2) = &\left(p^{2f_2} + p^{2f_2-2} + \ldots + 1\right)\sigma_3\langle f_1\rangle + p^{2f_2+2}(p^2+1)\sigma_3\langle f_1-1\rangle \\ &+ p^{2f_2+6}(p^2+1)\sigma_3\langle f_1-2\rangle + \ldots + p^{2f_2+4f_1-6}(p^2+1)\sigma_3\langle 1\rangle \\ &+ p^{2f_2+4f_1-2}\sigma_2\langle 1\rangle.\end{aligned}$$

Average orders

In [3] we evaluated $T(x) = \sum_{\det A \leq x} \tau_2^{(2)}(A)$, where A is in SNF. The absence of a zeta function does not allow satisfactory extension of this result.

Here we instead allow A to be in HNF and consider the following *zeta function* [5]:

$$Z_r(s) = \sum_{A \text{ in HNF},\ r\times r} (\det A)^{-s} = \sum \alpha_r(n)\, n^{-s}, \qquad s = \sigma + it, \quad \sigma > 1, \tag{10}$$

where $\alpha_r(n)$ is the HNF class-number, i.e., the number of $r \times r$ HNF matrices with determinant n.